Research Program

Sean Grate

Department of Mathematics and Statistics
Auburn University
Auburn, AL 36849
sean.grate@auburn.edu

October 31, 2024

My primary research interests lie in algebraic geometry, commutative algebra, and algebraic combinatorics. Problems which can be approached computationally are especially interesting as computations suggest a data-driven approach to theory-building via computing examples (and non-examples). When tackling problems in geometry and combinatorics, the problems typically involve very concrete data with which allow us to see what is going on "under the hood," and then the underlying algebraic structures yield effective means of proof.

Computing the Castelnuovo-Mumford regularity of toric varieties is an instance where this interplay comes to life as toric varieties are encoded with rich combinatorial data, but are a special instance of an algebraic-geometric object. I'm also interested in questions in algebraic combinatorics concerning the undimodality of sequences, and in a current project, am investigating the combinatorial structure of the stable Tamari poset. In my project on Lefschetz properties, in certain situations, the geometry of a set of points determined when their Artinian reduction had the weak Lefschetz property (WLP), but this then generalized to a more general statement about Betti diagrams and the WLP. For me, it's this ability to compute examples in very concrete terms and to then use that to guide more general statements that is paramount to enjoying mathematics.

1 Syzygies and toric varieties

Through the combinatorial lens, a toric variety is associated to a lattice with points in the lattice corresponding to sections of a line bundle. In the smallest case, i.e., that of monomial curves, L'vovsky showed that $\operatorname{reg}(I)$ is bounded by the sum of the two largest gaps of the semigroup of lattice points associated to I [15], giving a combinatorial interpretation of an algebraic invariant. In work in progress, I explore variants of this result to toric surfaces by explicitly computing $\operatorname{reg}(I)$ for a class of toric surfaces and showing that $\operatorname{reg}(I)$ is bounded by the lattice volume of the associated polygon [8].

The aforementioned class of toric surfaces can be described by the lattice points of the polygon associated to the toric variety. Given a polygon with integral vertices, we consider the toric variety defined only by the lattice points on the boundary of the polygon; in particular, *all* interior lattice points are left out. These examples lead to highly non-normal toric varieties, but surprisingly in many cases, whose regularity is small.

Using the sheaf-theoretic notion of regularity, the proofs amount to expressing sums of the interior lattice points as sums of the boundary lattice points. From this setup, the key ingredient is the combinatorics of the toric variety, so this work leads to the following project:

Project 1.1. *Given* any *toric surface* $\mathbf{V}(I)$*, bound* $\operatorname{reg}(I)$ *using only the combinatorial data of* I*.*

The outcome of this project would not only yield more concrete and computable bounds for the regularity, but would also shed light into how combinatorial data influences the behavior of I.

2 Lefschetz properties

In tomography, successive 2D scans of a 3D object are taken and patchworked together to get a picture of the 3D object. In a similar way, the landmark hard Lefschetz theorem provides a way to understand a complicated geometric object by slicing it with a hyperplane [13]. More precisely, for a smooth n-dimensional complex projective variety X, the cup product of the kth power of the cohomology class of a hyperplane yields an isomorphism between $H^{n-k}(X)$ and $H^{n+k}(X)$. A corollary of this is that multiplication by a generic linear form is injective up to degree n and surjective in higher degrees. This puts restrictions on the Hilbert function, and has thus been used with much success in the study of Artinian algebras with applications to geometric and enumerative combinatorics.

A (standard) graded *Artinian* ring A is a finite-dimensional graded vector space over $\mathbb{K}$, i.e., $A_d = 0$ for $d \gg 0$. Because of this, the Hilbert function of A can be regarded as a finite-dimensional vector, called the h-vector. Since elements of A vanish in sufficiently high degree, every element is a zero-divisor. When multiplication by ℓ is full rank in all degrees (i.e., either injective or surjective), we say A has the *weak Lefschetz property* (*WLP*).

Geometry of points

From a set of points $X \subset \mathbb{P}^n$ and with $R = \mathbb{K}[x_0, \dots, x_n]$, the Artinian reduction A_X is obtained by quotienting R/I_X by a generic linear form L; that is, $A_X = R/(\mathbf{I}(X), L)$. While the geometry of A_X seems to vanish, A_X still retains much of the information about X with the advantage of A_X now being Artinian. Motivated by this, joint with Hal Schenck [10], we investigated how the geometry of X influences failure or success of the WLP on A_X, leading to the following theorem:

Theorem 2.1. *Let $X_f \subset \mathbb{P}^n$ $(n \geq 3)$ be a set of distinct points lying on a hypersurface $\mathbf{V}(f)$ for some polynomial $f \in R$ with $\deg(f) = d$, such that $(\mathbf{I}(X_f))_d = (f)$ and $(\mathbf{I}(X_f))_{\leq d} = 0$. Choose $q \notin \mathbf{V}(f)$ such that if $X = X_f \cup \{q\}$, then $\mathbf{I}(X)_d = 0$ and $\dim_{\mathbb{K}}(\mathbf{I}(X)_{d+1}) = n$. Then the Artinian reduction A_X does not have the WLP. In particular, if ℓ is a general linear form, then $\cdot\ell \colon A_d \to A_{d+1}$ does not have full rank.*

The essence of this theorem is that if A_X is the Artinian reduction of a set of points lying on a *unique* hypersurface and one additional point lying off of it, then A_X does not have the WLP. Moreover, we can predict the degree in which the WLP fails as it fails in degree $\deg(f)$. The proof relies on the facts that $\mathbf{I}(X) = \mathbf{I}(X_f) \cap \mathbf{I}(\{q\})$ and that the initial degree of $\mathbf{I}(X)$ to explicitly write out the multiplication map $\cdot\ell \colon (A_X)_d \to (A_X)_{d+1}$.

Betti tables

Generalizing the results to Betti tables, since the majority of the points in X lie on a unique hypersurface, this led to a general pattern in the Betti tables of A_X, formalized via the following definition:

Definition 2.2. A Betti table B has an (n, d)-Koszul tail if it has an upper-left principal block of the form

$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & \dots & n-2 & n-1 & n \\
\hline
0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d-1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
d & 0 & n & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} & n & 1
\end{array}.$$

Note that this is only the top portion of the Betti table, where the dots represent zero-entries. If B has an (n, d)-Koszul tail and is the Betti table for an Artinian ring $\mathbb{K}[x_1, \dots, x_n]/I$, then we say B has a *maximal* (n, d)-Koszul tail.

For example, consider the pointset $X \subseteq \mathbb{P}^3$ given by the columns of the matrix below and the corresponding Betti table of A_X:

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \quad \mathrm{Betti}(A_X) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & 3 & 3 & 1 \\ 2 & . & 3 & 4 & 1 \\ 3 & . & . & 1 & 1 \end{array}.$$

$\mathrm{Betti}(A_X)$ has a maximal $(3,1)$-Koszul tail. Note that A_X satisfies the hypotheses of Theorem 2.1, so A_X does not have the WLP and the multiplication map $\cdot\ell \colon (A_X)_1 \to (A_X)_2$ is not full rank. The "Koszul-ness" of the linear strand of $\mathrm{Betti}(A_X)$ in the initial degree of A_X can act as an indicator for the WLP.

Theorem 2.3. *An Artinian algebra $A = \mathbb{K}[x_1, \ldots, x_n]/I$ whose Betti table has a maximal (n,d)-Koszul tail does not have the WLP; the map $\cdot\ell \colon A_d \to A_{d+1}$ is not full rank.*

A maximal Koszul tail is not sufficient for the WLP, however. The following example from [1] showcases two ideals with the same Betti table (which has a $(2,1)$-Koszul tail), but where one ideal has the WLP and the other does not:

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & . & . & . & . \\ 1 & . & 2 & 1 & . & . \\ 2 & . & 5 & 9 & 4 & . \\ 3 & . & 4 & 9 & 5 & . \\ 4 & . & . & 2 & 1 & . \\ 5 & . & . & . & . & 1 \end{array}$$

Ideal	Has the WLP
$(x_4^2, x_3x_4, x_3^3, x_2x_3^2 - x_2^2x_4, x_1x_3^2 - x_1x_2x_4 + x_2^2x_4, x_2^2x_3,$ $x_2^3, x_1^3x_4 - x_1^2x_2x_4 + x_1x_2^2x_4, x_1^3x_3, x_1^3x_2 - x_1^2x_2^2, x_1^4)$	True
$(x_1x_4, x_1^2, x_3x_4^2, x_2x_4^2, x_2^2x_4, x_1x_3^2,$ $x_1x_2^2 - x_3^2x_4, x_3^4, x_2x_3^3 - x_4^4, x_2^2x_3^2, x_2^4)$	False

Project 2.4. *What are sufficient conditions on the Betti table of A to determine if A has (or does not have) the WLP? Is the WLP visible purely from the Betti table alone? The Hilbert function is not sufficient to detect the WLP, as [16] and [18] find examples of level algebras that do not have the WLP.*

An interesting special case comes rom Stanley-Reisner theory, where a squarefree monomial ideal I_Δ is assigned to a simplicial complex Δ and then an Artinian reduction A_Δ to it. Tools such as Hochster's formula can again be leveraged to understand the Betti table of A_Δ. Combining this with the Koszul tail result, we can ask the following questions:

Project 2.5. *When does the Betti table of R/I_Δ have a Koszul tail? More generally, which Stanley-Reisner ideals have an Artinian reduction with (or without) the WLP?*

Constructions

The connected sum of two local Gorenstein rings was introduced by Ananthnarayan-Avramov-Moore [4], and their connections to Lefschetz properties by Iarrobino-McDaniel-Seceleanu [14]. In work originating from a workshop on Lefschetz properties in May 2023, we give finer information about these connected sums by explicitly computing the Betti numbers in terms of the factors in the connected sum, and show that the connected sum of doublings is itself a doubling [3]. With a description of the Betti numbers, one might expect that the information from the Betti table contains indicators for if the ring has the WLP, as in the case of the Koszul tails.

Project 2.6. *Find other constructions which preserve the WLP.*

In the case of [3], the fiber product and connected sum are formed over the ground field $\mathbb{K}$, so we know an explicit presentation of these constructions. For instance, with $R = \mathbb{K}[x_1, \ldots, x_m]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$, and with $A = R/I$ and $B = S/J$, the fiber product (over $\mathbb{K}$) is

$$A \times_{\mathbb{K}} B \cong \frac{\mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]}{I + J + \mathbf{x} \cap \mathbf{y}},$$

where $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. So, in the computation of the Betti numbers, $\mathbf{x} \cap \mathbf{y}$ only contributes in low degree. We leverage this and knowledge of the minimal free resolution of $\mathbf{x} \cap \mathbf{y}$ to compute the Betti numbers of the fiber product and connected sum. Hence, extending these results to fiber products and connected sums over more general algebras yields a more complete description of these constructions.

Project 2.7. *Compute the Betti numbers of the fiber product and the connected sum over arbitrary algebras which are compatible with the constructions.*

3 The combinatorics of the stable Tamari lattice

Stemming from an AMS MRC on algebraic combinatorics in 2024, I am working with Anna Pun, Herman Chau, Spencer Daugherty, and J. Carlos Martínez Mori to study and better understand the stable Tamari lattice and its posets [6]. In particular, given $\mathbf{b} \in \mathbb{N}^n$, we are interested in the properties of the *lower order ideal* of $\mathbf{b}$, denoted $\mathbf{b}_{\downarrow}$, and defined as the set of $\mathbf{a} \in \mathbb{N}^n$ such that $\mathbf{a} \leq_T \mathbf{b}$, where $\leq_T$ is the stable Tamari order. The stable Tamari lattice was introduced by Haiman through invariant polynomials [12], extended by Bergeron-Preville-Ratelle [5], and is a variation of the Tamari lattice introduced by Tamari [17]. It is described as *stable* since shifting $\mathbf{a}$ and $\mathbf{b}$ by the all-ones vector $\mathbf{1}^n = (1, 1, \ldots, 1)$ does not change the poset relation. As a running example, Figure 1 is the poset for $\mathbf{b} = (1, 2, 3)$.

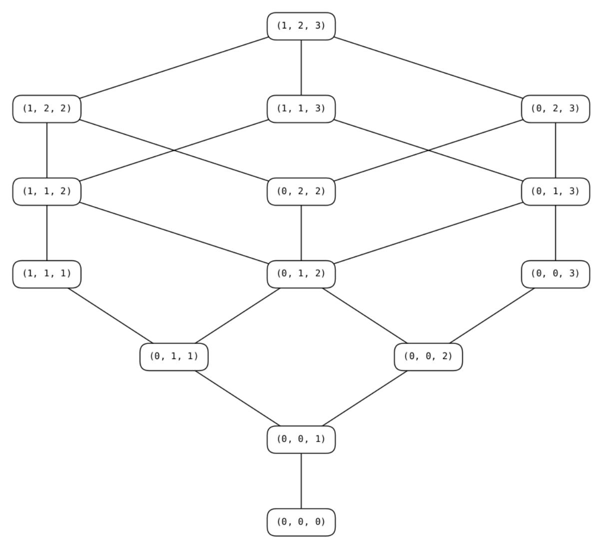

Figure 1: The Hasse diagram for $\mathbf{b} = (1, 2, 3)$.

The combinatorics of the stable Tamari lattice are still largely unexplored. In work in progress and supported by a large amount of computation, we have the following conjecture:

Conjecture 3.1. *For any $n \geq 1$, $\min\limits_{\mathbf{b} \in \mathbb{N}^n} \{|\mathbf{b}_{\downarrow}|\} = C_{n+1}$, where C_{n+1} is the $(n+1)$-st Catalan number. Moreover, the minimizers of this function are the elements $\mathbf{b} \in \mathbb{N}^n$ with exactly one peak.*

Conjecture 3.1 suggests a strong connection to parking functions, much-studied objects in the algebraic combinatorics community. As such, establishing an explicit combinatorial bijection to other Catalan objects is a natural goal.

To better understand the overall structure of $\mathbf{b}_{\downarrow}$, we define two generating functions

$$f_{\mathbf{b}}(t) = \sum_{\mathbf{a} \in \mathbf{b}_{\downarrow}} t^{\mathrm{ht}(\mathbf{a},\mathbf{b})} \quad \text{and} \quad F_{\mathbf{b}}(t) = \sum_{[\mathbf{p},\mathbf{q}] \subseteq \mathbf{b}_{\downarrow}} t^{\mathrm{ht}(\mathbf{p},\mathbf{q})},$$

where $\mathrm{ht}(\mathbf{p}, \mathbf{q})$ is the maximum length of a chain from $\mathbf{p}$ to $\mathbf{q}$ in $\mathbf{b}_{\downarrow}$. Continuing with the running example of $\mathbf{b} = (1, 2, 3)$, we get

$$f_{\mathbf{b}}(t) = t^6 + t^5 + 2t^4 + 3t^3 + 3t^2 + 3t + 1 \quad \text{and} \quad F_{\mathbf{b}}(t) = t^6 + 4t^5 + 8t^4 + 15t^3 + 21t^2 + 21t + 14.$$

Extensive computations support the following conjecture:

Conjecture 3.2. *For any $\mathbf{b} \in \mathbb{N}^n$, the coefficients of $f_{\mathbf{b}}(t)$ and $F_{\mathbf{b}}(t)$ are unimodal.*

From computation, we have also observed that the coefficients of $f_{\mathbf{b}}(t)$ and $F_{\mathbf{b}}(t)$ obey Macaulay's condition, so each is the Hilbert function of a $\mathbb{Z}$-graded algebra over a field [11]. With an affirmative answer to Project 3.3 below, one can hope that the associated Artinian algebras have the WLP, showing that the coefficients of $f_{\mathbf{b}}(t)$ and $F_{\mathbf{b}}(t)$ are unimodal.

Project 3.3. *Establish a connection between $f_{\mathbf{b}}(t)$ and an Artinian algebra (do the same for $F_{\mathbf{b}}(t)$). Show that the Artinian algebra has the WLP.*

4 Other Research

4.1 Discipline-Based Education Research

Joint with Melinda Lanius and Haile Gilroy, we developed the Course Analysis Alignment Tool (CAAT) to mathematically analyze the alignment of course objectives in classes [7]. Given a list of learning objectives, the CAAT has the instructor rank which objectives should be emphasized the most (ties are allowed), and then has them count how many times each objective appeared in an assignment, thus generating an observed ranking of objectives. The discrepancy or distance in these rankings is calculated and used to quantify how well an assignment aligns with the expectations of the instructor: the smaller the distance, the better the alignment (see Figure 2).

Alignment summaries are generated based on the quartiles of the distribution of distances. The first quartile (green) indicates an "Excellent" alignment, the second (yellow) an "Acceptable" score, and the last two (red) a "Poor" alignment. Using quartiles ensures that each rating category has an equal probability of being used with good ratings of "Excellent" and "Acceptable" accounting for half of the possible scores and "Poor" the other half.

When analyzing the impact of the CAAT on the perception of uniform homework, we found that the CAAT affected participants' definitions of homework quality. Especially with novice instructors, the CAAT shows promise as a professional development tool as it quantifies the alignment between proposed versus assessed learning objectives.

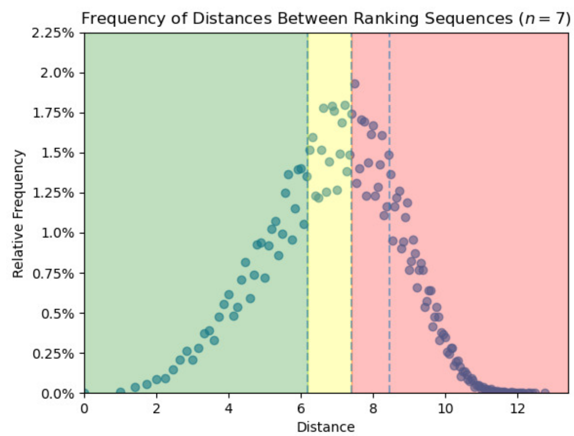

Figure 2: The distribution of distances on $n = 7$ learning objectives.

4.2 Contributions to Software

Much of my research is based on computations using software such as Python and Macaulay2. Through a data-driven approach, I compute many examples to help formulate conjectures and build evidence in tackling a research question. For instance, when approaching the WLP project with Koszul tails, generating many examples of Betti tables is what allowed us to really nail down that (1) the Koszul tail was a key ingredient, and (2) that *maximal* Koszul tails were the crux of the matter.

In my study of the regularity of toric varieties, I wrote code providing support for simplicial complexes, operations on them, and Hochster's formula, which included computing the rank of homology groups of a simplicial complex induced by a multigrading. I also wrote code for computations in topological data analysis (TDA), crucial for examining observed statistical phenomena and improved visualizations of the barcode/persistence diagrams.

At the Macaulay2 workshop in Minneapolis in June 2023, I worked with a team on developing the `MatrixSchubert` package for Macaulay2 [2]. It features methods for constructing and analyzing matrix Schubert varieties; these include specialized methods to compute the regularity of a quotient formed by a Schubert determinantal (or alternating sign matrix) ideal, reduced pipe dreams associated to a permutation, and more. This package has been incorporated into version 1.23 of Macaulay2.

Realizing a need for a standard `Permutation` class in Macaulay2, I have also independently developed the `Permutations` package to implement permutations and provide methods for them [9]. For example, this package allows users to use permutations to act on matrices, compute fundamental invariants such as the cycle type or the inversion set, and so on. This package will be included in version 1.24.11 of Macaulay2.

References

[1] Nancy Abdallah and Hal Schenck, *Free resolutions and Lefschetz properties of some Artin Gorenstein rings of codimension four*, Journal of Symbolic Computation **121** (2024), Paper No. 102257, 13. MR 4634957

[2] Ayah Almousa, Sean Grate, Daoji Huang, Patricia Klein, Adam LaClair, Yuyuan Luo, and Joseph McDonough, *The MatrixSchubert package for Macaulay2*, December 2023, arXiv:2312.07393 [math].

[3] Nasrin Altafi, Roberta Di Gennaro, Federico Galetto, Sean Grate, Rosa M. Miró-Roig, Uwe Nagel, Alexandra Seceleanu, and Junzo Watanabe, *Betti numbers for connected sums of graded Gorenstein artinian algebras*, Transactions of the AMS (2024), to appear.

[4] H. Ananthnarayan, Luchezar L. Avramov, and W. Frank Moore, *Connected sums of Gorenstein local rings*, Journal für die Reine und Angewandte Mathematik. [Crelle's Journal] **667** (2012), 149–176. MR 2929675

[5] François Bergeron and Louis-François Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets*, Journal of Combinatorics **3** (2012), no. 3, 317–341. MR 3029440

[6] Herman Chau, Spencer Daugherty, Sean Grate, J. Carlos Martínez Mori, and Anna Pun, *The combinatorics of stable Tamari posets*, in preparation.

[7] Haile Gilroy, Sean Grate, and Melinda Lanius, *Graph theoretic reflection to foster alignment in coordinated courses*, https://seangrate.com/misc/papers/CAAT.pdf, in revision (positive referee reports).

[8] Sean Grate, *Castelnuovo-Mumford regularity of toric surfaces*, in preparation.

[9] Sean Grate, *Permutations package for Macaulay2*, https://github.com/Macaulay2/M2/pull/3393, 2024.

[10] Sean Grate and Hal Schenck, *Betti Tables Forcing Failure of the Weak Lefschetz Property*, Lefschetz Properties, Springer INdAM Ser., vol. 59, Springer, Singapore, 2024, pp. 155–164. MR 4807665

[11] Mark Green, *Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann*, Algebraic curves and projective geometry (Trento, 1988), Lecture Notes in Math., vol. 1389, Springer, Berlin, 1989, pp. 76–86. MR 1023391

[12] Mark D. Haiman, *Conjectures on the quotient ring by diagonal invariants*, Journal of Algebraic Combinatorics. An International Journal **3** (1994), no. 1, 17–76. MR 1256101

[13] W. V. D. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, Cambridge; The Macmillan Company, New York, 1941. MR 3947

[14] Anthony Iarrobino, Chris McDaniel, and Alexandra Seceleanu, *Connected sums of graded Artinian Gorenstein algebras and Lefschetz properties*, Journal of Pure and Applied Algebra **226** (2022), no. 1, Paper No. 106787, 52. MR 4262073

[15] S. L'vovsky, *On inflection points, monomial curves, and hypersurfaces containing projective curves*, Mathematische Annalen **306** (1996), no. 4, 719–735. MR 1418349

[16] Juan C. Migliore, *The Geometry of the Weak Lefschetz Property and Level Sets of Points*, Canadian Journal of Mathematics **60** (2008), no. 2, 391–411.

[17] Dov Tamari, *The algebra of bracketings and their enumeration*, Nieuw Archief voor Wiskunde. Derde Serie **10** (1962), 131–146. MR 146227

[18] Fabrizio Zanello, *A non-unimodal codimension 3 level h-vector*, J. Algebra **305** (2006), no. 2, 949–956. MR 2266862