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Research Statement

Summary

My primary research interests lie in algebraic geometry, commutative algebra, and algebraic combinatorics. Much of this work is motivated by problems from enumerative combinatorics; here, one can think geometrically, prove algebraically, and compute combinatorially. I approach these problems via computation with software such as *Macaulay2* and Python.

Graded Möbius algebras. The graded Möbius algebra, arising in the recent proof of the Top-Heavy Conjecture [9], encodes the combinatorics of a matroid's lattice of flats. Work of LaClair, Mastroeni, McCullough, and Peeva characterizing which graphic matroids have Koszul graded Möbius algebras [26]. In recent work with McCullough, we study the Koszul property for the graded Möbius algebras of matroids associated to projective and affine geometries [20], and with a larger group of collaborators, the homological properties of these algebras [15].

Lefschetz properties. Lefschetz properties are a powerful tool to prove unimodality and log-concavity in enumerative combinatorics, as in the resolutions of the conjectures of Heron–Rota–Welsh and Mason–Welsh [2]. In the same spirit as the hard Lefschetz theorem, the weak and srong Lefschetz property can be defined for an Artinian algebra. With Schenck, I showed that Artinian algebras with a certain pattern in their Betti tables fail to have the weak Lefschetz property [21]. With other collaborators, I explicitly computed the graded Betti numbers of connected sums of Artinian Gorenstein algebras [4].

Toric varieties. My thesis extends L'vovsky's combinatorial bound on the Castelnuovo-Mumford regularity of monomial curves to toric surfaces [18]. For hollow toric surfaces, which use only the boundary lattice points of a polygon, the regularity can be arbitrarily large, yet it equals 2 for hollow triangles and squares, and I conjecture a general bound read off from the normal fan. With Betti, Holleben, and Salizzoni, I study incidence toric ideals, which encode the relations among three-point functions in conformal field theory, using Lefschetz theory, null designs, and pseudomanifolds [7].

The stable Tamari lattice. Parallel to this research, my collaborators and I introduced the stable Tamari lattice [10], stemming from work of Haiman on diagonal harmonics [22]. We determine its cover relations, height statistics, and the sizes of its lower order ideals [10, 11]. This project is meant to provide a combinatorial framework for problems extending from diagonal harmonics, which has already given rise to results such as the shuffle theorem and the generalized delta square conjecture.

Notation. Throughout, $R = \mathbb{K}[x_1, \ldots, x_n]$ and M is a finitely generated graded R-module with graded minimal free resolution $F_\bullet\colon F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_p \leftarrow 0$, where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$. The *graded Betti numbers* $\beta_{i,j}$ are arranged in the *Betti table* of M, whose entry in column i and row j is $\beta_{i,i+j}$, and the *Castelnuovo–Mumford regularity* (or simply *regularity*) $\operatorname{reg}(M) = \max\{j : \beta_{i,i+j} \neq 0\}$ is the index of its last nonzero row. The total Betti numbers $\beta_i = \sum_j \beta_{i,j}$ form the *Poincaré series* $P_M^R(t) = \sum_i \beta_i t^i$, and $\operatorname{HS}_M(t) = \sum_i \dim_{\mathbb{K}}(M_i)\, t^i$ is the *Hilbert series* of M.

1. The Graded Möbius Algebra of a Matroid

A matroid is a combinatorial abstraction of the notion of linear independence for vector spaces. Matroids have a rich combinatorial structure (they are simplicial complexes with an "exchange property"), and much work has been put into computing their enumerative statistics. Of note, the Dowling-Wilson Top-Heavy Conjecture asserted that the lattice of flats of a simple matroid is "top heavy" in the sense that if M is a simple matroid of rank r and W_i is the number of flats of rank i, then $W_i \leq W_{r-i}$ for all $0 \leq i \leq \frac{r}{2}$. This conjecture remained open for over 40 years until Huh and Wang resolved it when M is representable [23], and then Braden et al. proved it in full generality, in part, by using the combinatorics of the graded Möbius algebra $\mathsf{GMA}(M)$ (defined in [8]) and Lefschetz-type theorems of the Chow ring $\underline{\mathsf{CH}}(M)$ and the augmented Chow ring $\mathsf{CH}(M)$ [9].

1.1. Koszulness. A standard graded $\mathbb{K}$-algebra S is *Koszul* if $\mathbb{K}$ has a linear free resolution over S. Koszul algebras appear in many contexts, for example, as homogeneous coordinate rings of general canonical curves and of Schubert varieties. A standard result shows that if an algebra is Koszul, then it is quadratic, while the converse is false in general. Beyond constraining the graded minimal free resolution of $\mathbb{K}$ over S, the Koszul property imposes the identity $P^S_{\mathbb{K}}(t) \cdot \mathrm{HS}_S(-t) = 1$.

Mastroeni and McCullough showed that the Chow ring of a matroid is Koszul [29]. In general, the graded Möbius algebra need not be Koszul, though LaClair, Mastroeni, McCullough, and Peeva proved that a graph G is strongly chordal if and only if the graded Möbius algebra of the graphic matroid $M(G)$ is Koszul [26]. In forthcoming work [20], McCullough and I explore the Koszul property for projective and affine geometries.

Theorem 1.1 ([20]). *If M is modular (e.g., a projective geometry), then $\mathsf{GMA}(M)$ is Koszul. Equivalently, if M is a simple matroid and $\mathsf{GMA}\, M$ is Gorenstein, then $\mathsf{GMA}\, M$ is Koszul.*

Proving Koszulness in this setting uses techniques from [26] to exhibit a quadratic Gröbner basis. For affine geometries, this technique no longer works, and traditional techniques to show Koszulness such as constructing a Koszul filtration are elusive. However, we can still show that these are quadratic algebras and have evidence to believe some may be Koszul as well.

1.2. Betti Numbers. My collaborators and I have been working toward understanding the homological properties of these graded Möbius algebras. Using techniques from liaison theory, we have derived explicit formulas for all graded Betti numbers for the graded Möbius algebras of uniform matroids [15]. Since these algebras are combinatorially defined, we expect that the Betti numbers should have some combinatorial interpretation, too.

Project 1.2. *The graded Möbius algebra of a modular matroid is Gorenstein [20, 28], and thus is determined by the annihilator of the* basis polynomial *of the matroid [28]. Use the theory of inverse systems to compute the graded minimal free resolution of the graded Möbius algebra of a modular matroid.*

Generalizing projective and affine geometries, a *Sylvester* matroid is a matroid such that every pair of elements lies in a 3-circuit. We prove that if the graded Möbius algebra is quadratic and linearly presented, then the matroid is Sylvester [15]. We conjecture that if quadraticity is assumed, then the converse holds.

Project 1.3. *Let M be a simple matroid with at least 3 elements such that $\mathsf{GMA}(M)$ is quadratic. Prove that if M is Sylvester, then $\mathsf{GMA}(M)$ is linearly presented. What can be said about the Sylvester property and its relation (if any) to the Koszul property?*

A standard graded $\mathbb{K}$-algebra A is *Artinian* if it is a finite-dimensional graded vector space over $\mathbb{K}$, i.e., $A_d = 0$ for $d \gg 0$. When there exists a general linear form $\ell \in A_1$ such that the multiplication map $\times \ell^d : A_i \to A_{i+d}$ is full rank for all $d, i \geq 0$—that is, it is either injective or surjective—we say A has the *strong Lefschetz property* (*SLP*). If this map is full rank in all degrees for $d = 1$, then we say A has the *weak Lefschetz property* (*WLP*).

The SLP may be viewed as an algebraic stand-in for the landmark hard Lefschetz theorem: for a smooth n-dimensional complex projective variety X, the cup product of the k-th power of the class of a hyperplane section yields an isomorphism between $H^{n-k}(X)$ and $H^{n+k}(X)$. A corollary of this is that multiplication by a general linear form is injective up to degree n and surjective in higher degrees, a technique that has been used with great success in geometric combinatorics [9, 23, 34].

2.1. Betti Tables. Schenck and I investigated how the geometry of a set of points $X \subset \mathbb{P}^n$ influences whether A_X has the WLP, where A_X is the general Artinian reduction of the coordinate ring of X. We show that when all but one point lie on a unique degree-d hypersurface, A_X fails to have the WLP [21]. Then we generalize this to the following condition on the Betti table of A_X.

Definition 2.1. A Betti table B has an (n, d)-*Koszul tail* if its upper-left block has the form

	0	1	2	3	…	$n-2$	$n-1$	n
0	1	0	0	0	…	0	0	0
1	0	0	0	0	…	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$d-1$	0	0	0	0	…	0	0	0
d	0	n	$\binom{n}{2}$	$\binom{n}{3}$	…	$\binom{n}{n-2}$	n	1

Note that this is only the top portion of the Betti table. If B has an (n, d)-Koszul tail and is the Betti table for an Artinian algebra $\mathbb{K}[x_1, \ldots, x_n]/I$, then we say B has a *maximal* (n, d)-Koszul tail.

Theorem 2.2 ([21]). *An Artinian algebra $A = \mathbb{K}[x_1, \ldots, x_n]/I$ whose Betti table over $\mathbb{K}[x_1, \ldots, x_n]$ has a maximal (n, d)-Koszul tail does not have the WLP; the map $\times \ell \colon A_d \to A_{d+1}$ is not full rank for any $\ell \in A_1$.*

Project 2.3. *A maximal Koszul tail is not sufficient for the WLP, however, as [1, Theorem 1.6] shows. What conditions on the Betti table of A determine whether A has (or does not have) the WLP?*

2.2. Connected Sums. The connected sum $A_1 \#_T A_2$ of two local Gorenstein rings A_1 and A_2 over a local Cohen-Macaulay algebra T was introduced in [5], and its connections to Lefschetz properties were studied in [24]. This algebraic operation is motivated by the corresponding topological connected sum operation, as Artinian Gorenstein algebras serve as algebraic analogs of cohomology rings of complex, compact, connected, oriented manifolds. For instance, when M_1 and M_2 are manifolds (with the aforementioned assumptions) glued along a common submanifold N, then $A_1 \#_T A_2 \cong H^{2*}(M_1 \#_N M_2)$, where $A_i = H^{2*}(M_i, \mathbb{K})$ and $T \cong H^{2*}(N, \mathbb{K})$. My collaborators and I give finer information about these connected sums by explicitly computing the Betti numbers in terms of the factors in the connected sum, and we show that the connected sum of doubling—a construction from the theory of Gorenstein liaison—is itself a doubling [4], i.e., it is realized as the cokernel of a map from the doubling construction.

A finite set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ determines a *toric ideal* $I_\mathcal{A}$, the kernel of the map $x_i \mapsto \mathbf{t}^{\mathbf{a}_i}$, whose invariants are governed by the polytope $\mathrm{conv}(\mathcal{A})$ [13, 36]. My work asks how much of the homological structure of $I_\mathcal{A}$ the combinatorics of $\mathcal{A}$ can see.

3.1. Regularity of Toric Surfaces. The *Castelnuovo–Mumford regularity* (or simply, *regularity*) of R/I bounds the degrees of its defining equations and the complexity of Gröbner basis computations. Eisenbud and Goto conjectured that $\mathrm{reg}(I_X) \leq \deg X - \mathrm{codim}\, X + 1$ for every nondegenerate projective variety X [14]; McCullough and Peeva showed that this fails in general [30], but it remains open for toric varieties. For monomial curves, L'vovsky bounded the regularity by the sum of the two largest gaps between consecutive exponents [25]. Notably, this provides a *combinatorial* bound on regularity, an algebraic invariant.

For surfaces, Pinkham showed that if $X \subseteq \mathbb{P}^r$ is a non-degenerate projective surface of degree d and codimension r, then $\mathrm{reg}(X) \leq d - r + 4$ [31]. Lazarsfeld then showed that if X is also smooth, then $\mathrm{reg}(X) \leq d - r + 3$, and that this bound is optimal [27]. If X is toric, then $d = \deg(X)$ is the volume of the polytope associated to X, but this quantity is hard to compute in general. My thesis develops the analogous L'vovsky-style bound for toric surfaces [18]. In it, projections of toric surfaces are considered, amounting to omitting lattice points of a polygon P. The extreme case is the *hollow toric surface* of P, keeping only the lattice points on the boundary of P. In general, regularity does not behave well under projection, so one might expect the regularity of these toric surfaces to be quite large.

With too few boundary points, the regularity can be arbitrarily large: the quadrilateral with vertices $(0,0)$, $(d-1,d)$, $(d,d-1)$, (d,d) has no other boundary lattice points, and its hollow toric surface is a hypersurface of degree $2d$, so $\mathrm{reg}(R/I) = 2d - 1$, while its normalization has regularity 2. With enough boundary points, however, the regularity matches that of the normalization.

Theorem 3.1 ([18]). *Let $\triangle_k$ be the triangle with vertices $(0,0)$, $(k,0)$, $(0,k)$, and $\square_k = [0,k]^2$. For every $k \geq 3$, $\mathrm{reg}(R/I_{\triangle_k}) = 2$, and for every $k \geq 2$, $\mathrm{reg}(R/I_{\square_k}) = 2$.*

The proof rests on a combinatorial fact: for these polygons, every lattice point of dP with $d \geq 2$ is a sum of d boundary points of P. Hence $R/I_{\triangle_k}$ agrees with the coordinate ring of the k-th Veronese surface in every degree except 1, and a sheaf cohomology argument bounds its regularity by that of the Veronese surface. Smoothness alone does not force regularity 2; instead, the analogue of L'vovsky's gaps appears to be the lengths of the rays in the normal fan of P.

$$\mathrm{conv}\begin{pmatrix} 0 & d-1 & d & d \\ 0 & d & d-1 & d \end{pmatrix} \xrightarrow[\text{normal fan}]{\text{rays } \rho_i \text{ of}} \begin{pmatrix} -1 & 0 & -(d-1) & d-1 \\ 0 & -1 & d & d \end{pmatrix} \to \mathrm{reg}(R/I) = 2d-1 = \max_{\rho_i} ||\rho_i||$$

Conjecture 3.2. *If P is a smooth lattice polygon whose normal fan has rays $\rho_1, \ldots, \rho_r$, then its hollow toric surface satisfies $\mathrm{reg}(R/I) \leq \max\{2, \|\rho_1\|, \ldots, \|\rho_r\|\}$.*

Project 3.3. *Prove Conjecture 3.2. Since every smooth complete toric surface is obtained from $\mathbb{P}^2$ or a Hirzebruch surface by a sequence of blowups [13], induction on blowups and analyzing via sheaf cohomology is a natural strategy.*

3.2. Incidence Toric Ideals. Toric ideals also arise in physics: with Betti, Holleben, and Salizzoni [7], I study the algebraic relations among three-point functions in conformal field theory. For n massless particles with momenta $\mathbf{p}_i \in \mathbb{C}^d$, these are $C_{ijk} = (p_{ij}\, p_{jk}\, p_{ik})^{-1}$, where p_{ij} is the Lorentzian inner product of $\mathbf{p}_i$ and $\mathbf{p}_j$.

When $n \leq d$, their relations form a toric ideal.

For $t < k \leq n$, the *incidence toric ideal* $I_{n,k,t}$ is the kernel of the map sending a variable c_κ for each k-subset $\kappa \subseteq [n]$ to the product $\prod_{\tau \subseteq \kappa} x_\tau$ over its t-subsets; three-point functions give $I_{n,3,2}$. We applied results from the theory of Lefschetz properties to compute the dimension of $I_{n,k,t}$: up to a scalar, the exponent matrix is the matrix of $\times L^{k-t} \colon A_t \to A_k$, where $A = \mathbb{K}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ and $L = x_1 + \cdots + x_n$, so Stanley's theorem that A has the strong Lefschetz property [35] computes the dimension of $I_{n,k,t}$.

Theorem 3.4 ([7]). *The integer relations among the columns of the exponent matrix correspond to k-uniform null t-designs. Consequently, every nonzero element of $I_{n,k,t}$ has degree at least 2^t, and the associated polytope is $(2^t - 1)$-neighborly but not 2^t-neighborly.*

Balanced, orientable, normal pseudomanifolds without boundary give Graver basis elements of $I_{n,k,k-1}$: the octahedron gives a quartic, and $I_{6,3,2}$ is minimally generated by 15 octahedral quartics and 15 sextics.

Project 3.5. *Hyperoctahedral binomials generate an ideal $J_{n,k,t}$ with $I_{n,k,t} = J_{n,k,t} : (\prod_\kappa c_\kappa)^\infty$. Do $J_{n,k,t}$ and $I_{n,k,t}$ define varieties of the same degree, and how can $\deg I_{n,k,t}$ be computed efficiently? Beyond the toric range, where we resolved the case $d = n - 1$ with $3 \mid n$, describe the relations among three-point functions.*

4. The Stable Tamari Lattice

The classical Tamari lattice orders the parenthesizations of a product by directed applications of the associative law [37], and its realizations on binary trees, Dyck paths, and triangulations of polygons tie it to Catalan combinatorics [33]. Building on Haiman's work on diagonal harmonics [22], Bergeron and Préville-Ratelle introduced the r-Tamari lattices on r-Dyck paths in their study of higher trivariate diagonal harmonics [6], and Préville-Ratelle and Viennot extended these to ν-Tamari lattices with an arbitrary boundary path [32].

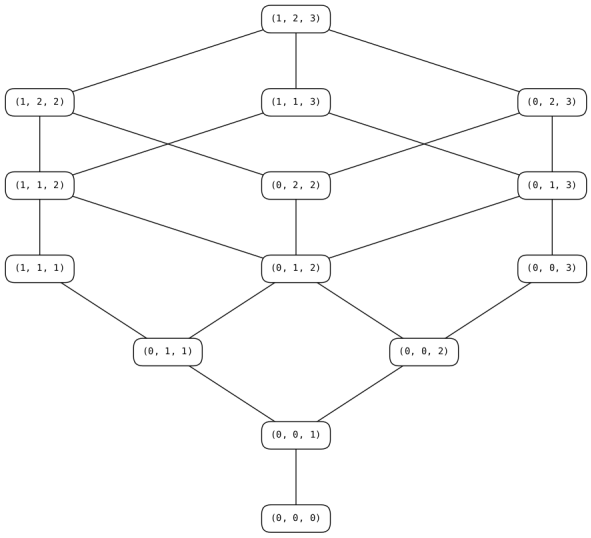


Figure 1: The Hasse diagram for $\mathbf{b}_\downarrow$ with $\mathbf{b} = (1, 2, 3)$.

With Chau, Daugherty, Haiman, Martínez Mori, and Pun, I introduced the *stable Tamari lattice* $\mathcal{ST}_n = (\mathbb{N}^n, \leq_T)$ [10], where $\mathbf{a} \leq_T \mathbf{b}$ if $\mathbf{a} \leq \mathbf{b}$ coordinatewise and $a_i - a_j \leq b_i - b_j$ whenever $i < j$ and $a_i, \ldots, a_{j-1} > a_j$. It is *stable* in that adding the same constant to every entry of both sequences preserves the order, and for a partition $\boldsymbol{\lambda}$, we have $\mathbf{a} \leq_T \boldsymbol{\lambda}$ if and only if $\boldsymbol{\lambda} - \mathbf{a}$ is a partition. For $\mathbf{b} \in \mathbb{N}^n$, the *downset* $\mathbf{b}_\downarrow$ is the set of $\mathbf{a} \in \mathbb{N}^n$ with $\mathbf{a} \leq_T \mathbf{b}$; as a running example, Figure 1 is the Hasse diagram of $\mathbf{b}_\downarrow$ for $\mathbf{b} = (1, 2, 3)$.

Theorem 4.1 ([10]).

1. *The r-Tamari lattices on r-Dyck paths of height $n + 1$ form a nested family of downsets whose union is $\mathcal{ST}_n$, and the downset of a partition is the ν-Tamari lattice of its boundary path.*
2. *Every element has exactly n upper covers. Writing $\mathbf{b} - \mathbf{a}$ in a basis of interval vectors determined by $\mathbf{a}$*

decides whether $\mathbf{a} \leq_T \mathbf{b}$*, and the sum of its coordinates is the maximum length of a chain from* $\mathbf{a}$ *to* $\mathbf{b}$*; the analogous basis for* $\mathbf{b}$ *gives the Möbius function, which takes values in* $\{0, \pm 1\}$*.*

3. $\mathcal{ST}_n$ *is semidistributive, and every interval admits an EL-labeling and is isomorphic to an interval in a classical Tamari lattice.*

Although $\mathcal{ST}_n$ is not graded, part (2) shows that $\sum_{\mathbf{b} \geq_T \mathbf{a}} t^{\mathrm{ht}(\mathbf{a},\mathbf{b})} = (1-t)^{-n}$ for every $\mathbf{a} \in \mathbb{N}^n$. Downsets are subtler: no closed formula for $|\mathbf{b}_\downarrow|$ is known, but fixing the last coordinate decomposes the downset of $(\mathbf{b}, k) \in \mathbb{N}^{n+1}$ into $k+1$ layers, each isomorphic to a downset in $\mathcal{ST}_n$, which gives a recursive formula for its size [10]. Among permutations, the sizes show a striking pattern.

Conjecture 4.2. *For any* $n \geq 1$, $\min_{\mathbf{b} \in \mathfrak{S}_n} \{|\mathbf{b}_\downarrow|\} = C_{n+1}$*, where* C_{n+1} *is the* $(n+1)$*-st Catalan number. Moreover, the minimizers of this function are the permutations with exactly one peak.*

Conjecture 4.2 suggests a strong connection to parking functions, much-studied objects in the algebraic combinatorics community. In forthcoming work, we establish a bijection between elements in $\mathbf{b}_\downarrow$ and lattice points in a polytope, and then show that unimodal permutations achieve this conjectured minimum [11].

To record heights within a downset, we define

$$f_{\mathbf{b}}(t) = \sum_{\mathbf{a} \leq_T \mathbf{b}} t^{\mathrm{ht}(\mathbf{a},\mathbf{b})} \quad \text{and} \quad F_{\mathbf{b}}(t) = \sum_{\mathbf{p} \leq_T \mathbf{q} \leq_T \mathbf{b}} t^{\mathrm{ht}(\mathbf{p},\mathbf{q})},$$

where $\mathrm{ht}(\mathbf{p}, \mathbf{q})$ is the maximum length of a chain from $\mathbf{p}$ to $\mathbf{q}$, so that $f_{\mathbf{b}}(1) = |\mathbf{b}_\downarrow|$. Continuing with the running example of $\mathbf{b} = (1, 2, 3)$, we get

$$f_{\mathbf{b}}(t) = t^6 + t^5 + 2t^4 + 3t^3 + 3t^2 + 3t + 1 \quad \text{and} \quad F_{\mathbf{b}}(t) = t^6 + 4t^5 + 8t^4 + 15t^3 + 21t^2 + 21t + 14.$$

Under the realization of the r-Tamari lattice as a downset, $f_{\mathbf{b}}(t)$ is exactly the interval generating function that Bergeron and Préville-Ratelle used in their conjectural formula for the Frobenius characteristic of higher trivariate diagonal harmonics [6]. We show that $\sum_{\mathbf{b} \in \mathbb{N}^n} f_{\mathbf{b}}(t)\, \mathbf{x}^{\mathbf{b}}$ is a finite sum of rational functions, one for each of the C_n Dyck paths of semilength n [10]. Computations support the following conjecture, and we prove the specialization of (1) at $t = 1$, namely $|\mathbf{b}_\downarrow| = |\mathrm{rev}(\mathbf{b})_\downarrow|$ [10].

Conjecture 4.3. *For every* $\mathbf{b} \in \mathbb{N}^n$*, (1)* $f_{\mathbf{b}}(t) = f_{\mathrm{rev}(\mathbf{b})}(t)$*, where* rev *reverses the entries, and (2)* $F_{\mathbf{b}}(t)$ *is unimodal.*

In contrast, $f_{\mathbf{b}}(t)$ need not be unimodal: the coefficients of t^8, t^9, t^{10} in $f_{(8,8,4,4)}(t)$ are $31, 30, 31$ [10].

5. Computation and Software

Computation drives much of my research: generating many Betti tables is what revealed the role of maximal Koszul tails, and experimental evidence on hollow polygons underlies Conjecture 3.2. Realizing a need for a standard permutation class in Macaulay2, I independently developed the `Permutations` package [17], included since version 1.24.11. I was part of the team behind the `MatrixSchubert` package [3], included since version 1.23, which constructs and analyzes matrix Schubert varieties, with specialized methods for the regularity of Schubert determinantal and alternating sign matrix ideals. I led the development of `YoungTableaux` [19], which we plan to merge into the `Tableaux` package [16], and helped develop `EquivariantEhrhart` [12], targeted for Macaulay2 version 1.26.11. I also contribute to the core system and review pull requests on its official repository.

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