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Research Statement

Overview

My primary research interest lie in **algebraic geometry**, **commutative algebra**, and **algebraic combinatorics**. Much of this work is motivated by problems from enumerative combinatorics; here, one can think geometrically, prove algebraically, and compute combinatorially. I approach these problems via computation with software such as *Macaulay2* and Python.

Many counting problems produce sequences of numbers that rise and then fall (**unimodal**), or that satisfy the stronger inequalities $a_k^2 \geq a_{k-1}a_{k+1}$ (**log-concave**). The binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ are the simplest example, but the same pattern appears in the coefficients of the chromatic polynomials of graphs [26, 36], in counts of independent sets of vectors, and throughout mathematics [38]. These patterns are easy to observe on a computer, yet are usually notoriously difficult to prove. One of the most powerful methods of proof is algebraic: realize the sequence as the dimensions of the graded pieces of an algebra, then show that multiplication by a well-chosen element is injective or surjective between consecutive pieces. This **Lefschetz** philosophy, inspired by the hard Lefschetz theorem from algebraic geometry, underlies Stanley's work on faces of polytopes [39] and the recent resolution of several decades-old conjectures about graphs and matroids [2, 10, 26], work recognized by June Huh's 2022 Fields Medal (see [6] for a survey).

A second theme is measuring the complexity of an algebra through its **syzygies**: its generators, the relations among them, the relations among those relations, and so on. These counts, the **Betti numbers**, are recorded in a **Betti table**, which acts as a quantitative summary of the algebra. When an algebra comes from combinatorics, this summary should carry combinatorial meaning; conversely, patterns in this summary can certify geometric and combinatorial properties. My research has four interrelated strands.

1. **Matroids and Koszul algebras.** The **graded Möbius algebra** of a matroid was a key ingredient in the recent proof of the decades-old Top-Heavy Conjecture. With McCullough, I study when these algebras are **Koszul**, the simplest possible behavior for syzygies [21]. With another group of collaborators, I compute their Betti numbers [17].
2. **Lefschetz properties.** With Schenck, I showed that a recognizable pattern in a Betti table forces an algebra to fail the **weak Lefschetz property** [22]. With other collaborators, I computed the Betti numbers of **connected sums**, an algebraic analogue of gluing manifolds together [4].
3. **Toric varieties.** My thesis extends a combinatorial bound on the **regularity** of monomial curves to toric surfaces [19]. With Betti, Holleben, and Salizzoni, I study **incidence toric ideals**, which arose from a question in theoretical physics [8].
4. **The stable Tamari lattice.** My collaborators and I introduced a new partial order on $\mathbb{N}^n$ extending the classical **Tamari lattice**, stemming from Haiman's work on **diagonal harmonics**, and we study its covers, chains, and downsets [12, 13].

1. Matroids and the Graded Möbius Algebra

A **matroid** is a combinatorial abstraction of linear independence [34]. The independent sets of the matroid might be the linearly independent subsets of a set of vectors, or the cycle-free sets of edges in a graph, or some other context where "independence" makes sense. A **flat** is a set closed under "span," and each flat has a **rank**. Matroids are a rich source of these unimodal and log-concave inequalities. For instance, if W_i is the number of flats of rank i in a simple matroid of rank r, Dowling and Wilson conjectured that $W_i \leq W_{r-i}$ whenever $i \leq r/2$. For finitely many points in the plane, not all on a line, this is the classical de Bruijn–Erdős theorem: the points determine at least as many lines as there are points. This **Top-Heavy Conjecture** stood for over 40 years, until Huh and Wang proved it for matroids coming from vectors (realizable matroids) [27] and Braden, Huh, Matherne, Proudfoot, and Wang proved it in general [10], in part by combining Lefschetz-type theorems for Chow rings of matroids with the combinatorics of the following algebra [9].

Definition 1.1. The **graded Möbius algebra** $\mathsf{GMA}(M)$ of a matroid M is the vector space with basis $\{y_F : F \text{ a flat of } M\}$ and multiplication $y_F \cdot y_G = y_{F \vee G}$ if $\mathrm{rk}(F \vee G) = \mathrm{rk}(F) + \mathrm{rk}(G)$, and $y_F \cdot y_G = 0$ otherwise, where $F \vee G$ is the smallest flat containing F and G.

This algebra has a grading via the rank function—the i-th piece of $\mathsf{GMA}(M)$ has dimension W_i—so the Top-Heavy Conjecture is a statement about the shape of this algebra. When every subset of $[n] = \{1, \ldots, n\}$ is independent, $\mathsf{GMA}(M)$ can be presented as $\mathsf{GMA}(M) \cong \mathbb{K}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$, with y_F corresponding to $\prod_{i \in F} x_i$.

1.1. Koszul Algebras. The algebra $\mathsf{GMA}(M)$ is generated by the elements y_p for the points p (flats of rank 1), and one can ask how complicated their relations must be. The simplest possible behavior is captured by the **Koszul** property: a standard graded algebra S is Koszul if the field $\mathbb{K}$ has a linear free resolution over S, so that the relations are quadratic and all the syzygies are linear, and the syzygies on the syzygies are linear, and so on. Koszulness implies nontrivial inequalities: if $\mathsf{GMA}(M)$ is Koszul, then the Betti numbers β_i of $\mathbb{K}$ over $\mathsf{GMA}(M)$ satisfy

$$\sum_{i \geq 0} \beta_i t^i = \frac{1}{W_0 - W_1 t + W_2 t^2 - \cdots + (-1)^r W_r t^r},$$

so the right-hand side must have nonnegative coefficients.

The Chow ring of every matroid is Koszul [32], but the graded Möbius algebra need not be: LaClair, Mastroeni, McCullough, and Peeva showed that for a graph G, $\mathsf{GMA}(M(G))$ is Koszul if and only if G is strongly chordal [30]. McCullough and I study geometries over finite fields. The **projective geometry** $\mathrm{PG}(n, q)$ is the matroid of lines through the origin in $\mathbb{F}_q^{n+1}$ (the prototypical **modular** matroid, i.e. one whose lattice of flats is modular) and the **affine geometry** $\mathrm{AG}(n, q)$ is the matroid of points of $\mathbb{F}_q^n$ under affine dependence.

Theorem 1.2 ([21])**.** *If M is modular (e.g., a projective geometry), then $\mathsf{GMA}(M)$ is Koszul. Equivalently, if M is simple and $\mathsf{GMA}(M)$ is **Gorenstein** (an algebraic form of Poincaré duality), then $\mathsf{GMA}(M)$ is Koszul.*

For affine geometries, the Gröbner basis techniques behind this theorem no longer work and Koszul filtrations have been elusive, but we can still show that these algebras are quadratic, and there is evidence that some are Koszul as well.

1.2. Betti Numbers and Sylvester Matroids. With a larger group of collaborators, I used liaison theory to find explicit formulas for all graded Betti numbers of the graded Möbius algebras of **uniform matroids**, which model vectors in general position [17]. Since these algebras are defined combinatorially, we expect their Betti numbers to have combinatorial interpretations. We also proved that if $\mathsf{GMA}(M)$ is quadratic and **linearly presented** (the relations among its defining quadrics are linear), then M is **Sylvester**: every line through two of its points contains a third point [17]. The Sylvester–Gallai theorem rules this out for points in the real plane, but over finite fields $\mathrm{PG}(n,q)$ is always Sylvester, and $\mathrm{AG}(n,q)$ is Sylvester when $q > 2$.

Project 1.3. *The graded Möbius algebra of a modular matroid is Gorenstein, so it is determined by a single polynomial, the* basis polynomial *of the matroid [21, 31]. Use Macaulay's theory of* ***inverse systems*** *to compute the graded minimal free resolutions of these algebras.*

Project 1.4. *Let M be a simple matroid with at least three points such that $\mathsf{GMA}(M)$ is quadratic. Prove that if M is Sylvester, then $\mathsf{GMA}(M)$ is linearly presented, and determine what relation, if any, the Sylvester property has to Koszulness.*

2. Lefschetz Properties

Sperner's theorem states that the largest family of subsets of $[n]$ in which no set contains another has $\binom{n}{\lfloor n/2 \rfloor}$ members. A beautiful algebraic proof uses the algebra B_n, whose basis is indexed by the subsets of $[n]$. Multiplication by $L = x_1 + \cdots + x_n$ sends a subset to the sum of all subsets obtained by adding one new element (Figure 1). Stanley showed that multiplication by every power L^j, from degree i to degree $i + j$, has full rank, i.e., is injective or surjective [40]; Sperner's theorem and the unimodality of the binomial coefficients follow quickly.

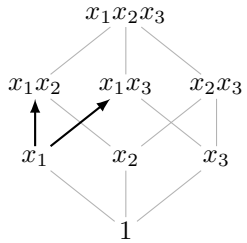


Figure 1: The basis of B_3. The arrows show $L \cdot x_1 = x_1x_2 + x_1x_3$.

More generally, a graded $\mathbb{K}$-algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_s$ is **Artinian** if it is a finite-dimensional $\mathbb{K}$-vector space. It has the **strong Lefschetz property** (**SLP**) if there is a linear form ℓ such that multiplication by $\ell^j \colon A_i \to A_{i+j}$ has full rank for all i and j, and the **weak Lefschetz property** (**WLP**) if this holds for $j = 1$. A major corollary of The WLP is that it forces the dimensions of the A_i to be unimodal. The names come from the hard Lefschetz theorem in algebraic geometry, a tool used with great success in combinatorics [10, 27, 39], and deciding which algebras have these properties is subtle: even for complete intersections, whether the WLP always holds in characteristic zero is a well-known open problem [25].

2.1. Betti Tables. A natural source of Artinian algebras is finite sets of points: for $X \subset \mathbb{P}^n$, the algebra of polynomial functions on X can be cut down to an Artinian algebra A_X. Schenck and I showed that if all but one point of X lie on a unique hypersurface of degree d, then A_X fails the WLP [22]. The underlying reason is visible in the Betti table, whose column i records the i-th syzygies and whose rows record their degrees. In these examples, rows 1 through $d-1$ of the table are empty, and row d contains the binomial coefficients of the Koszul complex:

	0	1	2	$\cdots$	$n-1$	n
0	1	0	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
d	0	n	$\binom{n}{2}$	$\cdots$	n	1

We call this an (n, d)-**Koszul tail**, and call it *maximal* when it occurs in the Betti table of an Artinian algebra in n variables.

Theorem 2.1 ([22]). *If the Betti table of an Artinian algebra $A = \mathbb{K}[x_1, \ldots, x_n]/I$ has a maximal (n, d)-Koszul tail, then A fails the WLP: the multiplication map $\ell \colon A_d \to A_{d+1}$ does not have full rank for any linear form ℓ.*

Project 2.2. *Koszul tails cannot characterize the WLP [1, Theorem 1.6]. Which features of the Betti table of an Artinian algebra determine whether it has the WLP?*

In related work on Betti tables, my collaborators and I studied **connected sums** of Artinian Gorenstein algebras, the algebraic counterpart of gluing manifolds together [5, 28]. We computed the Betti numbers of a connected sum in terms of its pieces, and showed that a connected sum of **doublings** (a construction from liaison theory) is again a doubling [4].

3. Toric Varieties

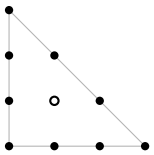
A lattice polygon P determines a surface: each lattice point of P gives a monomial, and together these monomials parametrize the **toric surface** of P, whose equations (the **toric ideal**) encode the integer relations among the lattice points. For instance, the corners of the unit square give the monomials $1, s, t, st$, and the relation $(0,0)+(1,1) = (1,0)+(0,1)$ makes the toric surface the quadric $x_0x_3 = x_1x_2$ in $\mathbb{P}^3$. More generally, any finite set $\mathcal{A} \subset \mathbb{Z}^d$ determines a toric ideal $I_{\mathcal{A}}$ whose invariants are governed by the polytope $\operatorname{conv}(\mathcal{A})$ [15, 41], and my work asks how much of the homological structure of $I_{\mathcal{A}}$ the combinatorics of $\mathcal{A}$ can see.

3.1. Regularity of Toric Surfaces. The **Castelnuovo–Mumford regularity** of R/I, the index of the last nonzero row of its Betti table, bounds the degrees of the defining equations and the complexity of Gröbner basis computations. Eisenbud and Goto conjectured that $\operatorname{reg}(I_X) \leq \deg X - \operatorname{codim} X + 1$ for every nondegenerate projective variety X [16]. McCullough and Peeva showed that this fails in general [33], but it remains open for toric varieties. For **monomial curves**, which are projections of the rational normal curve, L'vovsky bounded the regularity by the sum of the two largest gaps between consecutive exponents [29]. My thesis develops the analogous story for surfaces [19], where omitting lattice points of a polygon projects its toric surface. The extreme case, the **hollow toric surface** of P, keeps only the lattice points on the boundary of P.

With too few boundary points, the regularity can be arbitrarily large: the quadrilateral with vertices $(0,0)$, $(d, d-1)$, (d, d), $(d-1, d)$ has no other boundary lattice points, and its hollow toric surface is a hypersurface of degree $2d$, so $\operatorname{reg}(R/I) = 2d - 1$, while its normalization has regularity 2. With enough boundary points, however, the regularity matches that of the normalization.

Theorem 3.1 ([19]). *Let $\triangle_k$ be the triangle with vertices $(0,0)$, $(k,0)$, $(0,k)$, and $\square_k = [0,k]^2$. For every $k \geq 3$, $\operatorname{reg}(R/I_{\triangle_k}) = 2$, and for every $k \geq 2$, $\operatorname{reg}(R/I_{\square_k}) = 2$.*

The proof rests on a combinatorial fact: for these polygons, every lattice point of dP with $d \geq 2$ is a sum of d boundary points of P. Hence $R/I_{\triangle_k}$ agrees with the coordinate ring of the k-th Veronese surface in every degree except 1, and a sheaf cohomology argument bounds its regularity by that of the Veronese surface (Figure 2). Smoothness alone does not force regularity 2 [19]. Instead, the analogue of L'vovsky's



	0	1	2	3	4	5	6	7	8
0	1	.	.	.	.	.	.	.	.
1	.	17	43	36	8	.	.	.	.
2	.	.	10	55	100	83	37	9	1

Figure 2: The boundary lattice points of $\triangle_3$ and the Betti table of $R/I_{\triangle_3}$: 17 quadrics, and regularity 2.

gaps appears to be the **normal fan** of P, which records the primitive outer normal vectors $\rho_1, \ldots, \rho_r$ of its edges: for the example with lattice points $(0,0)$, $(d-1,d)$, $(d,d-1)$, and (d,d) from before, the normal vector $(-(d-1),d)$ has lattice-length $2d-1$ (the largest length of a ray in the normal fan), exactly the regularity.

Conjecture 3.2. *If P is a smooth lattice polygon whose normal fan has rays $\rho_1, \ldots, \rho_r$, then its hollow toric surface satisfies* $\operatorname{reg}(R/I) \leq \max\{2, \|\rho_1\|, \ldots, \|\rho_r\|\}$.

Project 3.3. *Prove Conjecture 3.2. Since every smooth complete toric surface is obtained from $\mathbb{P}^2$ or a Hirzebruch surface by a sequence of blowups [15], induction on blowups and analyzing via sheaf cohomology is a natural strategy.*

3.2. Incidence Toric Ideals. Toric ideals also arise in physics: with Betti, Holleben, and Salizzoni [8], I study the algebraic relations among **three-point functions** in conformal field theory. For n massless particles with momenta $\mathbf{p}_i \in \mathbb{C}^d$, these are $C_{ijk} = (p_{ij}\, p_{jk}\, p_{ik})^{-1}$, where p_{ij} is the Lorentzian inner product of $\mathbf{p}_i$ and $\mathbf{p}_j$; when $n \leq d$, their relations form a toric ideal.

Definition 3.4. For $t < k \leq n$, the **incidence toric ideal** $I_{n,k,t}$ is the kernel of the map sending a variable c_κ for each k-subset $\kappa \subseteq [n]$ to the product $\prod_{\tau \subseteq \kappa} x_\tau$ over its t-subsets. Three-point functions give $I_{n,3,2}$.

The exponent matrix records which t-subsets lie in which k-subsets, so up to a scalar it is the matrix of multiplication in the algebra $B_n = \mathbb{K}[y_1, \ldots, y_n]/(y_1^2, \ldots, y_n^2)$ by $L^{k-t}\colon (B_n)_t \to (B_n)_k$, and Stanley's theorem that B_n has the SLP computes the dimension of $I_{n,k,t}$.

Theorem 3.5 ([8])**.** *The integer relations among the columns of the exponent matrix correspond to* ***null t-designs****, integer weightings of the k-subsets of $[n]$ in which every t-subset is covered with total weight zero. Consequently, every nonzero element of $I_{n,k,t}$ has degree at least 2^t, and the associated polytope is (2^t-1)-neighborly but not 2^t-neighborly.*

Balanced, orientable, normal pseudomanifolds without boundary give Graver basis elements of $I_{n,k,k-1}$; the octahedron gives a quartic, and $I_{6,3,2}$ is minimally generated by 15 octahedral quartics and 15 sextics.

Project 3.6. *Hyperoctahedral binomials generate an ideal $J_{n,k,t}$ with $I_{n,k,t} = J_{n,k,t} : (\prod_\kappa c_\kappa)^\infty$. Do $J_{n,k,t}$ and $I_{n,k,t}$ define varieties of the same degree, and how can* $\deg I_{n,k,t}$ *be computed efficiently? Beyond the toric range, where we resolved the case $d = n-1$ with $3 \mid n$, describe the relations among three-point functions.*

4. The Stable Tamari Lattice

The Catalan numbers $1, 1, 2, 5, 14, 42, \ldots$ count over two hundred families of combinatorial objects [37], including the ways to parenthesize a product of $n+1$ factors. The **Tamari lattice** [42] orders these

parenthesizations by applying the associative law $(ab)c \to a(bc)$. Its generalizations, the r-Tamari [7] and ν-Tamari [35] lattices, arose in the study of **diagonal harmonics**, a space of polynomials in two sets of n variables whose dimension, $(n+1)^{n-1}$, is the number of **parking functions** [23, 24], and whose finer structure led to the shuffle theorem of Carlsson and Mellit [11].

With Chau, Daugherty, Haiman, Martínez Mori, and Pun, I introduced the **stable Tamari lattice** $\mathcal{ST}_n = (\mathbb{N}^n, \leq_T)$ [12], where $\mathbf{a} \leq_T \mathbf{b}$ if $\mathbf{a} \leq \mathbf{b}$ coordinatewise and $a_i - a_j \leq b_i - b_j$ whenever $i < j$ and $a_i, \ldots, a_{j-1} > a_j$. The r-Tamari lattices sit inside $\mathcal{ST}_n$ as a nested family of downsets whose union is all of $\mathcal{ST}_n$, and for a partition $\boldsymbol{\lambda}$, we have $\mathbf{a} \leq_T \boldsymbol{\lambda}$ if and only if $\boldsymbol{\lambda} - \mathbf{a}$ is a partition. We showed that every element has exactly n upper covers and every interval is isomorphic to an interval in a classical Tamari lattice [12].

For $\mathbf{b} \in \mathbb{N}^n$, the **downset** $\mathbf{b}_\downarrow$ is the set of all $\mathbf{a}$ with $\mathbf{a} \leq_T \mathbf{b}$ (Figure 3). No closed formula for $|\mathbf{b}_\downarrow|$ is known, but downsets decompose recursively into downsets of vectors with fewer entries [12], and permutations of $[n]$ show a striking pattern.

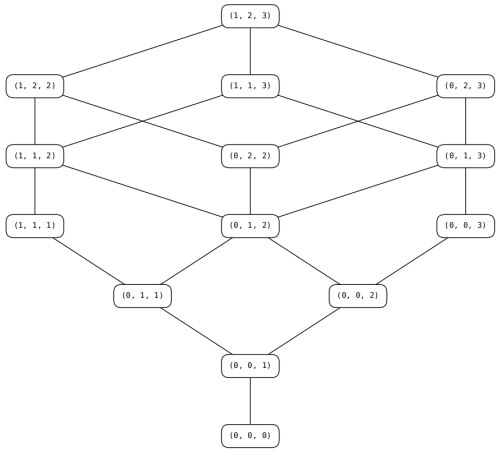


Figure 3: The Hasse diagram of $\mathbf{b}_\downarrow$ for $\mathbf{b} = (1,2,3)$, which has 14 elements.

Conjecture 4.1. *For every $n \geq 1$, $\min_{\mathbf{b} \in \mathfrak{S}_n} |\mathbf{b}_\downarrow| = C_{n+1}$, the $(n+1)$-st Catalan number, and the minimizers are exactly the **unimodal permutations** (those with a single peak).*

For example, $\mathbf{b} = (1,2,3)$ is unimodal and $|\mathbf{b}_\downarrow| = 14 = C_4$, and the conjecture suggests a strong connection to parking functions. In forthcoming work, a bijection between $\mathbf{b}_\downarrow$ and the lattice points of a polytope shows that unimodal permutations achieve the conjectured minimum [13].

To record heights within a downset, we define

$$f_{\mathbf{b}}(t) = \sum_{\mathbf{a} \leq_T \mathbf{b}} t^{\mathrm{ht}(\mathbf{a},\mathbf{b})} \quad \text{and} \quad F_{\mathbf{b}}(t) = \sum_{\mathbf{p} \leq_T \mathbf{q} \leq_T \mathbf{b}} t^{\mathrm{ht}(\mathbf{p},\mathbf{q})},$$

where $\mathrm{ht}(\mathbf{p},\mathbf{q})$ is the length of a longest chain from $\mathbf{p}$ to $\mathbf{q}$, so that $f_{\mathbf{b}}(1) = |\mathbf{b}_\downarrow|$. For $\mathbf{b} = (1,2,3)$, $f_{\mathbf{b}}(t) = t^6 + t^5 + 2t^4 + 3t^3 + 3t^2 + 3t + 1$ and $F_{\mathbf{b}}(t) = t^6 + 4t^5 + 8t^4 + 15t^3 + 21t^2 + 21t + 14$. When $\mathbf{b}_\downarrow$ is an r-Tamari lattice, $f_{\mathbf{b}}(t)$ is exactly the interval generating function in Bergeron and Préville-Ratelle's conjectural formula for higher trivariate diagonal harmonics [7].

Conjecture 4.2. *For every $\mathbf{b} \in \mathbb{N}^n$, $f_{\mathbf{b}}(t) = f_{\mathrm{rev}(\mathbf{b})}(t)$, where* rev *reverses the entries, and the coefficients of $F_{\mathbf{b}}(t)$ are unimodal.*

We prove the first part at $t = 1$, namely $|\mathbf{b}_\downarrow| = |\mathrm{rev}(\mathbf{b})_\downarrow|$ [12]. In contrast, $f_{\mathbf{b}}(t)$ itself need not be unimodal: in $f_{(8,8,4,4)}(t)$, the coefficients of t^8, t^9, t^{10} are $31, 30, 31$ [12].

5. Computation and Software

Computation lies at the heart of most of my projects. In Macaulay2, I developed `Permutations` [18], led `YoungTableaux` [20], helped develop `MatrixSchubert` [3] and `EquivariantEhrhart` [14], and contribute to the core system by reviewing pull requests and contributing code.

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