Algebra Seminar: Tropical Algebra

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These notes are heavily borrowed and inspired from *Invitation to Nonlinear Algebra* by Michałek and Sturmfels.

1 Motivating Examples

Theorem 1. Let G be a weighted directed graph on n nodes with adjacency matrix D_G. The entry of the matrix $D_G^{\odot(n-1)}$ in row i and column j is the length of a shortest (Hamiltonian?) path from node i to node j in the graph G.

Example 1. A company has n jobs and n workers and wants to assign each job to one and only one worker. Let x_{ij} be the cost of assigning job i to worker j. Naturally, the company wishes to find the cheapest assignment $\pi \in \mathfrak{S}_n$:

$$\min \left\{ x_{1\pi(1)} + x_{2\pi(2)} + \ldots + x_{n\pi(n)} \ : \ \pi \in \mathfrak{S}_n \right\}.$$

Theorem 2. The minimum from Example 1 is the tropical determinant of the matrix $X = (x_{ij})$. The tropical determinant solves the assignment problem.

2 What is tropical algebra?

Definition 1. The *tropical semiring* $(\overline{R}, \oplus, \odot)$ (also called the *min-plus algebra*) is the set $\overline{R} = \mathbb{R} \cup \{\infty\}$, where ∞ represents plus-infinity, and

$$x \oplus y \coloneqq \min(x, y) \quad \text{and} \quad x \cdot y \coloneqq x + y.$$

Remark 1. Infinity is the additive identity and zero is the multiplicative identity, i.e.

$$x \oplus \infty = x \quad \text{and} \quad x \odot 0 = x.$$

Furthermore, we have

$$x \odot \infty = \infty \quad \text{and} \quad x \oplus 0 = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}.$$

Example 2.

$$\begin{aligned} 2 \oplus 5 &= \min\{2,5\} = 2 \\ 6 \odot 3 &= 6 + 3 = 9 \\ 5 \odot 3 \oplus 2 \odot 4 &= \min\{5+3, 2+4\} = \min\{8,6\} = 6 \end{aligned}$$

Theorem 3.

- Pascal's triangle under tropical addition consists of all zeros.
- The binomial theorem holds.
- The Freshman's Dream holds.

Example 3.

$$\begin{aligned} (x \oplus y)^{\odot 3} &= x^{\odot 3} \oplus x^{\odot 2} \odot y \oplus x \odot y^{\odot 2} \oplus y^{\odot 3} \\ (x \oplus y)^{\odot 3} &= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\ &= 3\min\{x, y\} \\ &= \min\{3x, 2x+y, x+2y, 3y\} \\ &= \min\{3x, 3y\} \\ &= x^{\odot 3} \oplus y^{\odot 3}. \end{aligned}$$

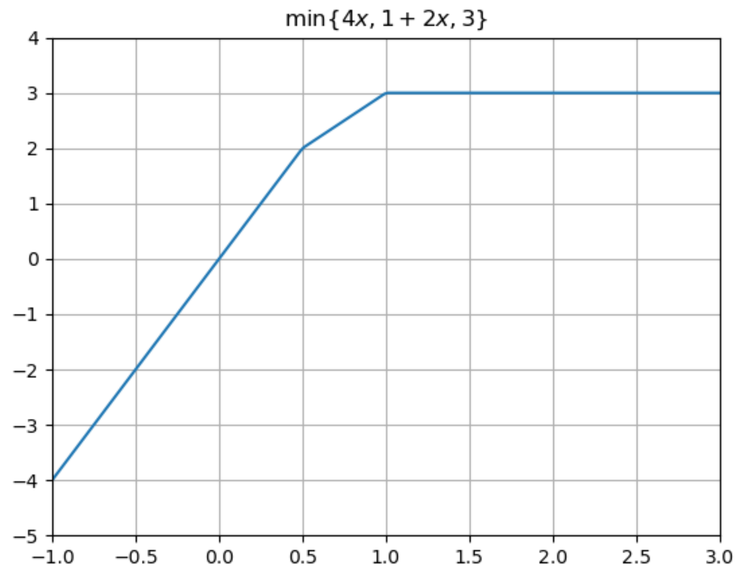
Remark 2. The tropical semiring should remind one of logarithms. Note that for small positive real numbers, we have $\log(u \cdot v) = \log(u) \odot \log(v)$ and $\log(u+v) \approx \log(u) \oplus \log(v)$. Tropical geometry thus pops up when drawing log-log plots in $\mathbb{R}^2_>$, suggesting connections to statistical models.

Definition 2. A *tropical polynomial* is a function that is the minimum of finitely many affine-linear functions. A real number u is said to be a *tropical root* of a given tropical polynomial if that minimum is attained at least twice when the affine-linear functions are evaluated at the argument u.

Example 4. Consider the tropical polynomial given by

$$\begin{aligned} \operatorname{trop}(f)(u) &= u^{\odot 4} \oplus 1 \odot u^{\odot 2} \oplus 3 \\ &= \min\{4u, 1+2u, 3\}. \end{aligned}$$

Then the roots of $\operatorname{trop}(f)$ are $u = 1$ and $u = \frac{1}{2}$.



3 Linear Algebra

Note 1. We can do matrix and vector operations over the tropical semiring. Consider the case in $\mathbb{R}^3$ with vectors $v, w \in \mathbb{R}^3$:

$$v^\top w = v_1 \odot w_1 \oplus v_2 \odot w_2 \oplus v_3 \odot w_3 = \min\{v_1 + w_1, v_2 + w_2, v_3 + w_3\};$$
$$vw^\top = \begin{bmatrix} v_1 \odot w_1 & v_1 \odot w_2 & v_1 \odot w_3 \\ v_2 \odot w_1 & v_2 \odot w_2 & v_2 \odot w_3 \\ v_3 \odot w_1 & v_3 \odot w_2 & v_3 \odot w_3 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 & v_1 + w_2 & v_1 + w_3 \\ v_2 + w_1 & v_2 + w_2 & v_2 + w_3 \\ v_3 + w_1 & v_3 + w_2 & v_3 + w_3 \end{bmatrix}.$$

Example 5. Consider a weighted directed graph G where each directed edge (i, j) has an associated (non-negative) weight d_{ij}. If (i, j) is not present in G, then we set $d_{ij} = \infty$. In this way, we get the adjacency matrix $D_G = (d_{ij})$ of G. Note that this need not be symmetric. We now return to one of our motivating examples from the beginning.

Theorem 4. Let G be a weighted directed graph on n nodes with adjacency matrix D_G. The entry of the matrix $D_G^{\odot(n-1)}$ in row i and column j is the length of a shortest (Hamiltonian?) path from node i to node j in the graph G.

Proof. Let $d_{ij}^{(r)}$ denote the minimum length of any path from node i to node j using at most r edges in G. Clearly, $d_{ij}^{(1)} = d_{ij}$ for any i, j. Since the d_{ij} are non-negative, for any i, j, there exists a shortest path from i to j that visits each node of G at most once. So the length of a shortest path from i to j equals $d_{ij}^{(n-1)}$.
We can then see that there is a recursive relationship when finding the length of a shortest path:

$$d_{ij}^{(r)} = \min\left\{d_{ik}^{(r-1)} + d_{kj} \ : \ k = 1, 2, \ldots n\right\}.$$

We may rewrite this as

$$\begin{aligned} d_{ij}^{(r)} &= d_{i1}^{(r-1)} \odot d_{1j} \oplus d_{i2}^{(r-1)} \odot d_{2j} \oplus \ldots \oplus d_{in}^{(r-1)} \odot d_{nj} \\ &= \begin{bmatrix} d_{i1}^{(r-1)} & d_{i2}^{(r-1)} & \vdots & d_{in}^{(r-1)} \end{bmatrix} \begin{bmatrix} d_{1j} \\ d_{2j} \\ \vdots \\ d_{nj} \end{bmatrix}. \end{aligned}$$

It follows from induction on r that $d_{ij}^{(r)}$ equals the entry in row i and column j of $D_G^{\odot r}$ since the RHS is the tropical product of row i in $D_G^{\odot(r-1)}$ and column j in D_G. Applying this to $r = n-1$, we get that $d_{ij}^{(n-1)}$ is the entry in row i and column j of $D_G^{\odot(n-1)}$. □

Definition 3. Let $X = (x_{ij})$ be an $n \times n$ matrix with entries in $\mathbb{R} \cup \{\infty\}$. The *tropical determinant* is defined in the expected way:

$$\operatorname{tropdet}(X) \coloneqq \bigoplus_{\pi \in \mathfrak{S}_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \ldots \odot x_{n\pi(n)}.$$

Returning to our previous example with the company assigning n jobs to exactly n workers, the tropical determinant solves the assignment problem.

Theorem 5. The tropical determinant solves the assignment problem.

Proof. Let x_{ij} be the cost of assigning job i to worker j and set $X = (x_{ij})$. The company wants to minimize the total cost of assigning jobs to workers. Thus we have that such a minimum is

$$\begin{aligned} \min\left\{x_{1\pi(1)} + x_{2\pi(2)} + \ldots + x_{n\pi(n)} \ : \ \pi \in \mathfrak{S}_n\right\} &= \bigoplus_{\pi \in \mathfrak{S}_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \ldots \odot x_{n\pi(n)} \\ &= \operatorname{tropdet}(X). \end{aligned}$$

□

Note 2. Although the tropical determinant is at least $n!$ computations, the polynomial-time *Hungarian algorithm* computes $\operatorname{tropdet}(X)$.