Algebra Seminar: Tropical Algebra

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These notes are heavily borrowed and inspired from *Introduction to Tropical Geometry* by Maclagan and Sturmfels.

1 What is tropical algebra?

Definition 1. The *tropical semiring* $(\overline{R}, \oplus, \odot)$ (also called the *min-plus algebra*) is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, where ∞ represents plus-infinity, and

$$x \oplus y \coloneqq \min(x, y) \quad \text{and} \quad x \cdot y \coloneqq x + y.$$

Remark 1. Infinity is the additive identity and zero is the multiplicative identity, i.e.

$$x \oplus \infty = x \quad \text{and} \quad x \odot 0 = x.$$

Furthermore, we have

$$x \odot \infty = \infty \quad \text{and} \quad x \oplus 0 = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}.$$

Example 1.

$$\begin{aligned} 2 \oplus 5 &= \min\{2, 5\} = 2 \\ 6 \odot 3 &= 6 + 3 = 9 \\ 5 \odot 3 \oplus 2 \odot 4 &= \min\{5 + 3, 2 + 4\} = \min\{8, 6\} = 6 \end{aligned}$$

Remark 2. The tropical semiring should remind one of logarithms. Note that for small positive real numbers, we have $\log(u \cdot v) = \log(u) \odot \log(v)$ and $\log(u + v) \approx \log(u) \oplus \log(v)$ (whenever $\oplus = \max$). Tropical geometry thus pops up when drawing log-log plots in $\mathbb{R}^2_>$, suggesting connections to statistical models.

Definition 2. A *tropical polynomial* is a function that is the minimum of finitely many affine-linear functions. A real number u is said to be a *tropical root* of a given tropical polynomial if that minimum is attained at least twice when the affine-linear functions are evaluated at the argument u.

Definition 3. Let p be a tropical polynomial. Define the *hypersurface* $V(p)$ of p to be the set of all points $\mathbf{w} \in \mathbb{R}^n$ at which this minimum is attained at least twice. In other words, $\mathbf{w} \in V(p)$ if and only if p is not linear at $\mathbf{w}$.

Example 2. Let $n = 1$ and consider the polynomial

$$p(x) = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d.$$

Assume that $b - a \leq c - b \leq d - c$. Then

$$V(p) = \{b - a, c - b, d - c\}.$$

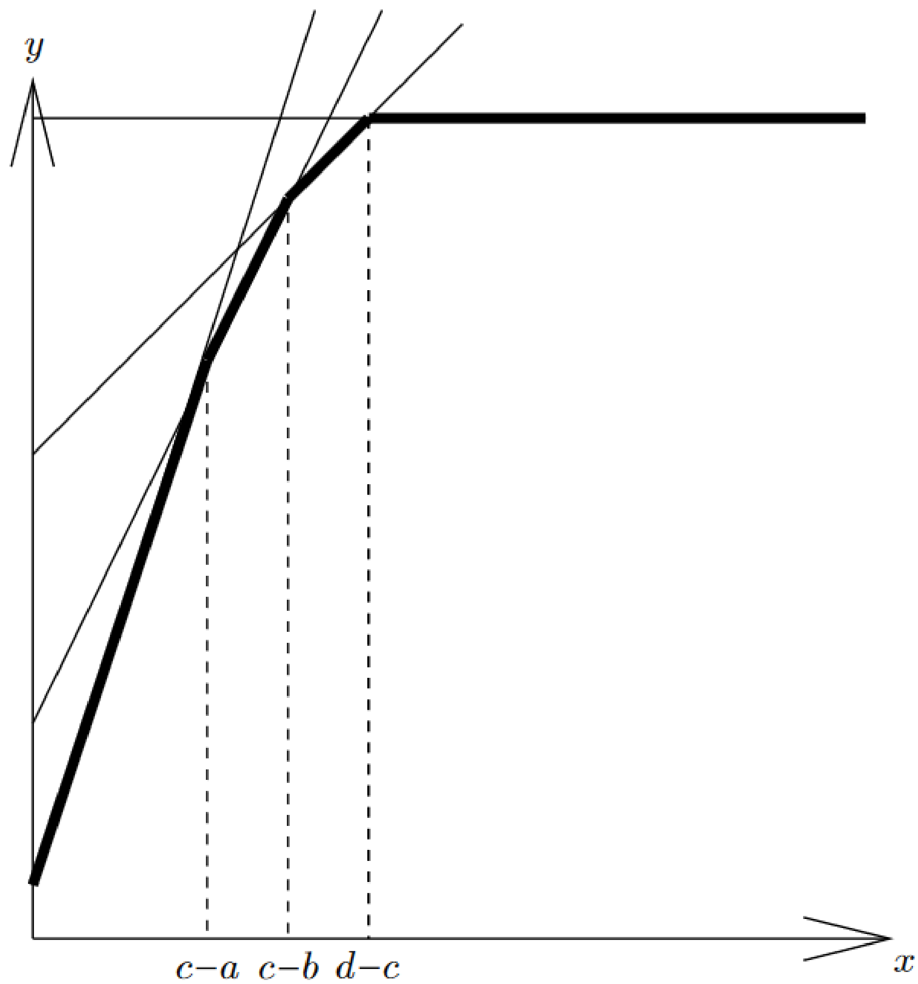

Figure 1.1.1. The graph of a cubic polynomial and its roots.

2 Plane Curves

Definition 4. Consider the polynomial

$$p(x,y) = \bigoplus_{(i,j)} c_{i,j} \odot x^i \odot y^j.$$

The corresponding hypersurface $V(p)$ is a *plane tropical curve.*

Proposition 1. The curve $V(p)$ is a finite graph that is embedded in the plane $\mathbb{R}^2$. It has both bounded and unbounded edges, all edge slopes are rational, and this graph satisfies a *balancing condition* around each node.

Example 3. Consider the polynomial

$$p(x,y) = a \odot x \oplus b \odot y \oplus c.$$

Then $V(p)$ consists of all points (x,y) where the function

$$p\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto \min(a+x, b+y, c)$$

is not linear. It consists of three half-rays emanating from the point $(x,y) = (c-a, c-b)$: What happens if the lines intersect in special position, e.g. two half-rays line up? The notion

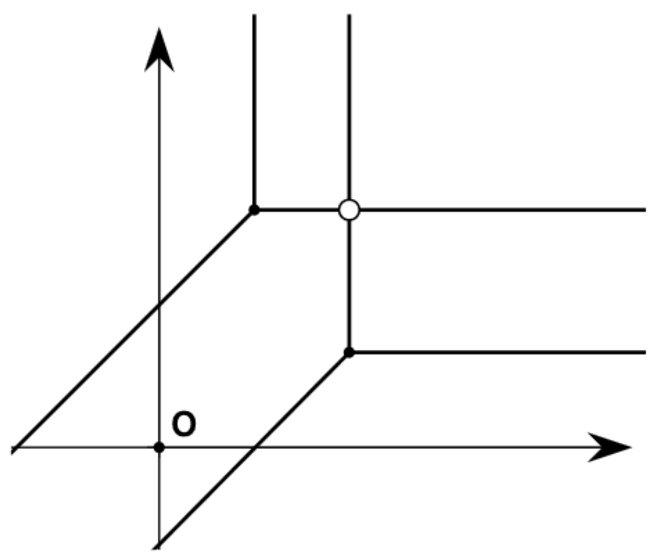

Figure 1.3.1. Two lines in the tropical plane meet in one point.

of stable intersection is used to get a unique intersection point.

Example 4. Consider the general quadratic polynomial

$$p(x, y) = a \odot x^2 \oplus b \odot xy \oplus c \odot y^2 \oplus d \odot y \oplus e \oplus f \odot x.$$

Assume that

$$b + f < a + d \quad d + f < b + e \quad b + d < c + f.$$

Then the graph of $p \colon \mathbb{R}^2 \to \mathbb{R}$ is the lower envelope of six planes in $\mathbb{R}^3$.

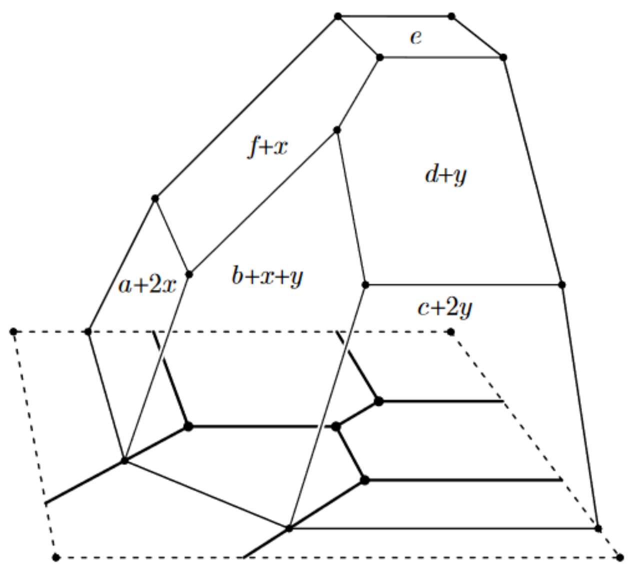

Figure 1.3.2. The graph and the curve defined by a quadratic polynomial.

2.1 Newton Polygons

Definition 5. Let $p(x, y)$ be a polynomial in two variables, in either classical or tropical arithmetic. Its *Newton polygon* Newt (p) is defined as the convex hull in $\mathbb{R}^2$ of all points (i, j) such that $x^i y^j$ appears in the expansion of $p(x, y)$.

Note 1. If $p(x, y)$ is a tropical polynomial, then its curve $V(p)$ can be constructed from Newt (p). Namely, the planar dual to $V(p)$ is a subdivision of Newt (p) into smaller polygons. If the smaller triangles are triangles, then the subdivision is a *triangulation*. The triangulation is *unimodular* if each cell is a lattice triangle of unit area $\frac{1}{2}$. In this case, $V(p)$ is a *smooth tropical curve.*

Example 5. The curve on the left is smooth and is an example of a *tropcical elliptic curve*. The curve on the right is not smooth.

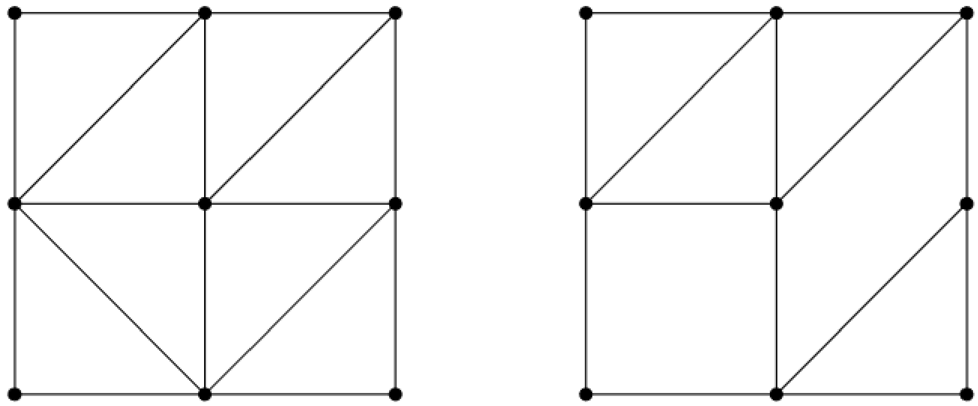

Figure 1.3.3. Two subdivisions of the Newton polygon of a biquadratic curve. Their planar duals are the curves in Figure 1.3.4.

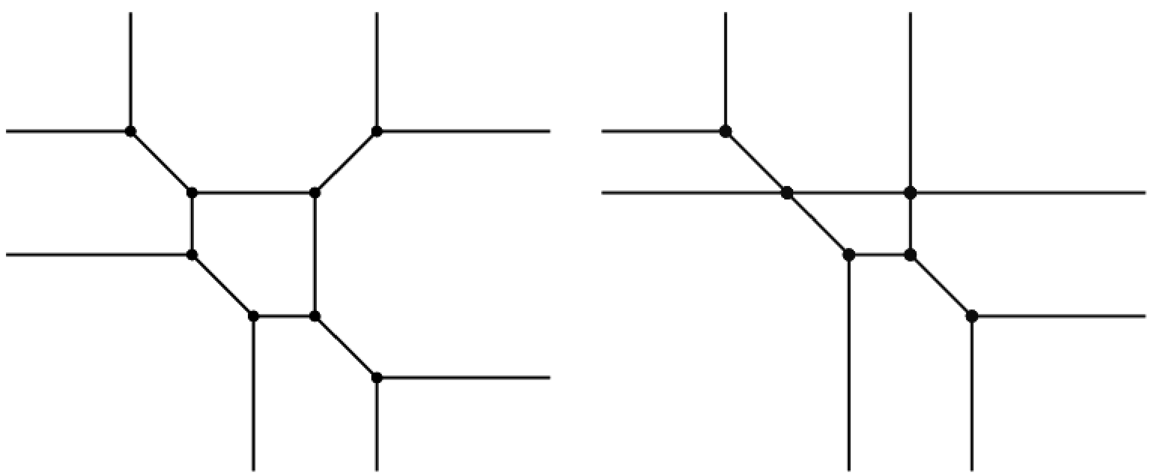

Figure 1.3.4. Two tropical biquadratic curves. The curve on the left is smooth.

3 Bezout's Theorem

Note 2.

- Two general lines meet in one point.
- Two general points lie on a unique line.
- A general line and quadric meet in two points.
- Two general quadrics meet in four points.

- Five general points lie on a unique quadric.

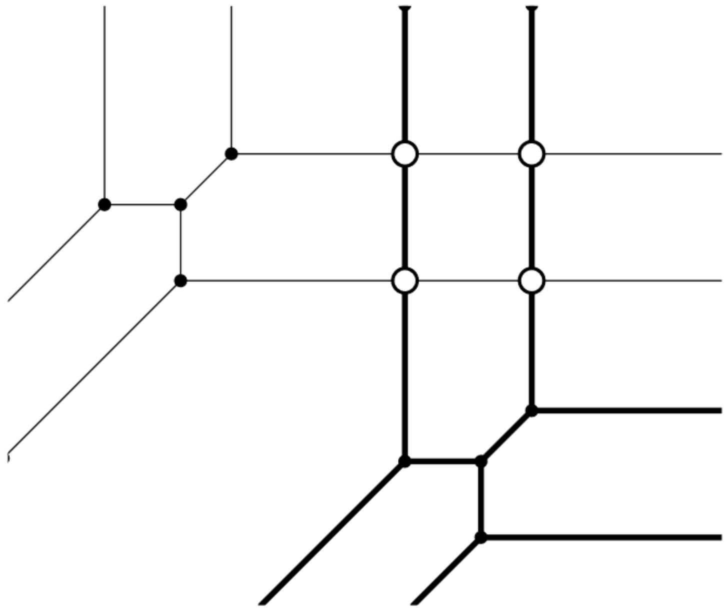

Figure 1.3.5. Bézout's Theorem: Two quadratic curves meet in four points.

Definition 6. Tropical curves whose Newton polygons are the standard triangles with vertices $(0,0),(0,d),(d,0)$ are called *curves of degree* d. A curve of degree d has d rays (counting multiplicities) perpendicular to each of the three edges of its Newton polygon (triangle).

Theorem 1. Consider two tropical curves C and D of degree c and d in $\mathbb{R}^2$. If the two curves intersect transversally, then the number of intersection points, counted with multiplicities, is equal to cd.

Note 3. We can drop the transverse condition if we introduce the notion of stable intersection.

Theorem 2. The limit of the point configuration $C_\epsilon \cap D_\epsilon$ is independent of the choice of perturbations. It is a well-defined multiset of cd points contained in the intersection $C \cap D$.

Theorem 3. Any two curves of degrees c and d in $\mathbb{R}^2$ intersect stably in a well-defined mutliset of cd points.

Example 6.

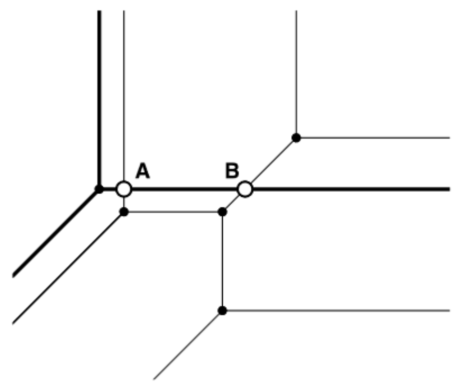

Figure 1.3.6. The stable intersection of a line and a quadric.

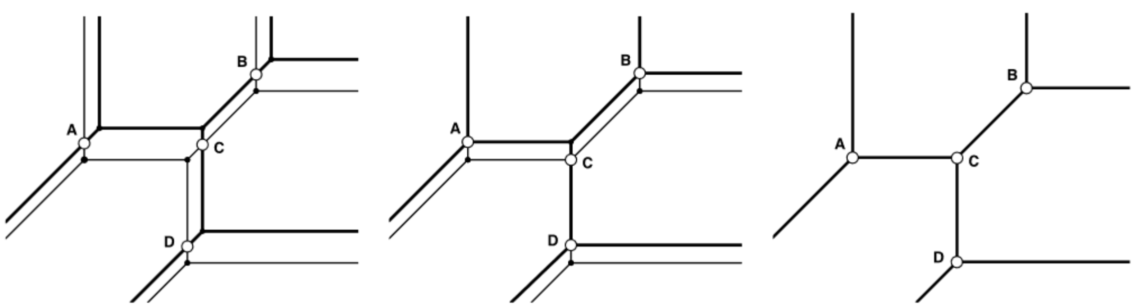

Figure 1.3.7. The stable intersection of a quadric with itself.