Gorenstein Graded Möbius Algebras are Koszul

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April 18, 2026

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The Crew

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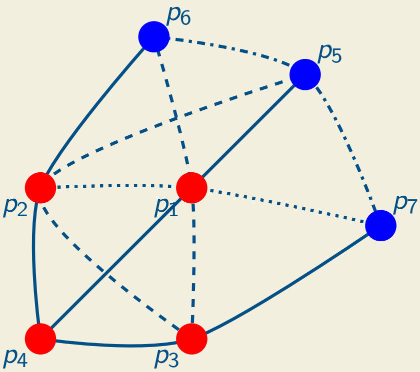
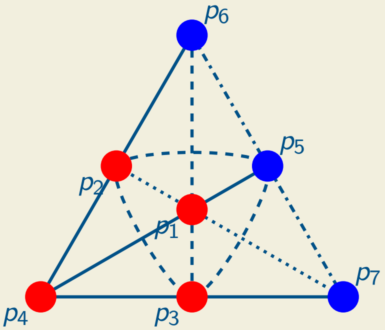
Projective Geometries

$$
\begin{array}{ccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7
\end{array}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
$$





Matroids Crash Course

Definition

A matroid $M = (E, \mathcal{I}$ consists of a ground set E and independent sets $\mathcal{I} \subseteq 2^E$ such that

(1) $\emptyset \in \mathcal{I}$;

(2) If $I' \subseteq I$ and $I \in \mathcal{I}$, then $I' \in \mathcal{I}$;

(3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

Example

- Vector spaces
- Cycles of a graph
- Hyperplane arrangements
- Boolean lattices

Matroids Crash Course

Let $M = (E, \mathcal{I})$ be a matroid.

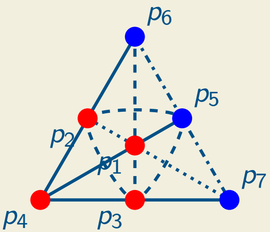
- Elements of $\mathcal{I}$ are called independent sets; maximal ones are called bases.
- Sets not in $\mathcal{I}$ are called dependent sets; minimal ones are called circuits.
- The rank of a set $F \subseteq E$ is the cardinality of a maximal independent set in F. The closure of a set $F \subseteq E$ is

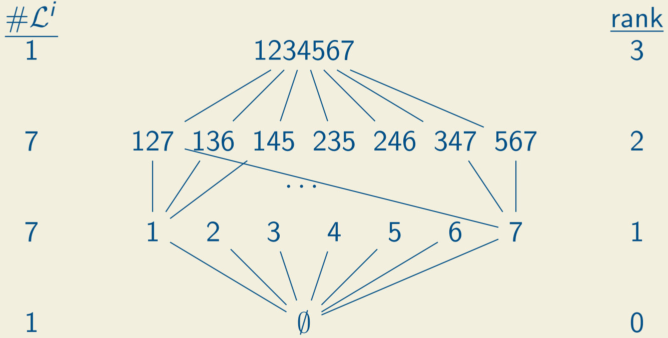
$$\overline{F} := \{e \in E \ : \ \mathrm{rank}(F \cup \{e\}) = \mathrm{rank}(F)\} .$$

  A set F such that $\overline{F} = F$ is called a flat.

The Lattice of Flats

$$
\begin{array}{ccccccc}
\color{red}{p_1} & \color{red}{p_2} & \color{red}{p_3} & \color{red}{p_4} & \color{blue}{p_5} & \color{blue}{p_6} & \color{blue}{p_7}
\end{array}
$$
$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
$$





Geometric Lattices

Definition

A *lattice* *is a poset with well-defined join* $\vee$ *and meet* $\wedge$ *operations.*

Definition

A lattice is called *geometric* *if it is*

(1) atomic

(2) (upper) semimodular

$$\text{rank}(F \vee G) + \text{rank}(F \wedge G) \leq \text{rank}(F) + \text{rank}(G)$$

$$\dim(U + V) + \dim(U \cap V) \leq \dim(U) + \dim(V)$$

Theorem

A lattice is geometric if and only if it is the lattice of flats of a matroid.

How do different ranks compare?

Theorem (Greene, 1970)

Let $\mathcal{L}$ be a geometric lattice with $\operatorname{rank}\mathcal{L} = n > 1$, and with

$$\mathcal{L}^k := \{F \in \mathcal{L} \ : \ \operatorname{rank}(F) = k\},$$

let $w(k) = \left|\mathcal{L}^k\right|$. Then

(1) $w(1) \leq w(k)$ for all $k = 2, \ldots, n-1$; and

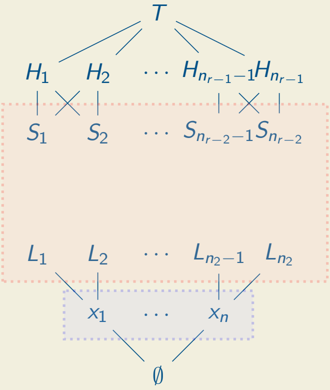
(2) $w(1) = w(k)$ if and only if $k = n-1$ and L is modular.

$$\operatorname{rank}(F \vee G) + \operatorname{rank}(F \wedge G) = \operatorname{rank}(F) + \operatorname{rank}(G)$$

The Top-Heavy Conjecture

Theorem (Huh-Wang, 2017; Braden-Huh-Matherne-Proudfoot-Wang, 2020)

Let $\mathcal{L}$ be a geometric lattice of rank r. For any $k \leq j \leq r - k$, there is an injective map $\iota\colon \mathcal{L}^k \rightharpoonup \mathcal{L}^j$.



Graded Möbius Algebras

Definition

Let $\mathcal{M}$ be a simple matroid with lattice of flats $\mathcal{L}(M)$. The graded Möbius algebra of M is the standard graded $\mathbb{K}$-algebra

$$\mathrm{B}_M = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K} x_F,$$

with multiplication

$$x_F x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rank}(F \vee G) = \operatorname{rank}(F) + \operatorname{rank}(G), \\ 0 & \text{otherwise} \end{cases}.$$

How can you force a Betti table into a talk?

Fact

Let M be a simple matroid with lattice of flats $\mathcal{L}(M)$.

(1) The Hilbert function of B_M is the rank-generating function of $\mathcal{L}(M)$

(2) $\operatorname{reg} \mathrm{B}_M = \operatorname{rank} M = \operatorname{rank} \mathcal{L}(M)$

(3) $\operatorname{pdim} \mathrm{B}_M = |E(M)|$

Example

Let $M = \mathrm{PG}(2,2)$. It is a rank 3 matroid on 7 elements with rank generating function $1 + 7t + 7t^2 + t^3$. The Betti table for B_M is

	0	1	2	3	4	5	6	7
0	1	.	.	.	.	.	.	.
1	.	21	64	70	14	.	.	. .
2	.	.	.	14	70	64	21	.
3	.	.	.	.	.	.	.	1

When is a $\mathbb{K}$-algebra Koszul?

Theorem (Conca-Rossi-Valla, 2001)

Suppose R is a graded Gorenstein $\mathbb{K}$-algebra presented by quadrics. If $\operatorname{reg}(R) \leq 2$ *or if* $\operatorname{reg}(R) = 3$ *and* $\operatorname{codim}(R) \leq 4$, *then R is Koszul.*

Theorem (Mastroeni-Schenck-Stillman, 2021/2023)

For all integers $c \geq r + 2 \geq 6$, there exists a graded, quadratic, Gorenstein $\mathbb{K}$-algebra R with $\operatorname{reg}(R) = r$ *and* $\operatorname{codim}(R) = c$ *that is not Koszul.*

Theorem (Roos, 1993)

Koszulness of a $\mathbb{K}$-algebra cannot definitively be checked by computing finitely many steps of the minimal free resolution.

When is B_M Koszul?

Theorem (Laclair-Mastroeni-McCullough-Peeva, 2025)

Let $\mathcal{M}$ be a simple matroid. There is a presentation of B_M:

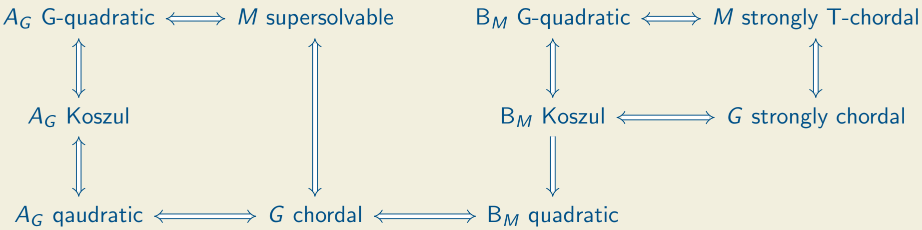
$$\mathrm{B}_M \cong \frac{\mathbb{K}[y_i : \ i \in E(M)]}{(y_i^2 : \ i \in E) + (y_{C\setminus i} - y_{C\setminus j} : \ C \in \mathcal{C}(M))}$$

Theorem (Laclair-Mastroeni-McCullough-Peeva, 2025)

Let M be the cycle matroid of a graph G. Then B_M is Koszul if and only if G is strongly chordal.

Orlik-Solomon + Graded Möbius Algebra Duality

Let $M = M(G)$ be the cycle matroid of a graph G, and let A_M be its Orlik-Solomon algebra and B_M its graded Möbius algebra.



Projective Geometries

Theorem (Greene, 1970; Maeno-Numata, 2016)

Let M be a simple matroid and $\mathcal{L}(M)$ be its lattice of flats. The following are equivalent:

1. *(1) $\mathcal{L}(M)$ is modular.*
2. *(2) The rank-generating function of $\mathcal{L}(M)$ is symmetric.*
3. *(3) The Hilbert function of B_M is symmetric.*
4. *(4) B_M is Gorenstein.*

$$\mathrm{B}_M \cong \frac{\mathbb{K}[y_i : i \in E(M)]}{Ann\left(\sum_{B \in \mathcal{B}(M)} y_B\right)}$$

Projective Geometries

Theorem (Grate-McCullough, 2026+)

If M is a projective geometry, then B_M is Koszul.

$M = \mathrm{PG}(d, q) \xRightarrow[\text{[GM26+]}]{} M$ chordal $\xRightarrow[\text{[LMMP25]}]{\text{Proposition 3.5(a)}} \mathrm{B}_M$ quadratic

$M = \mathrm{PG}(d, q) \xRightarrow{\text{[GM26+]}} M$ strongly T-chordal

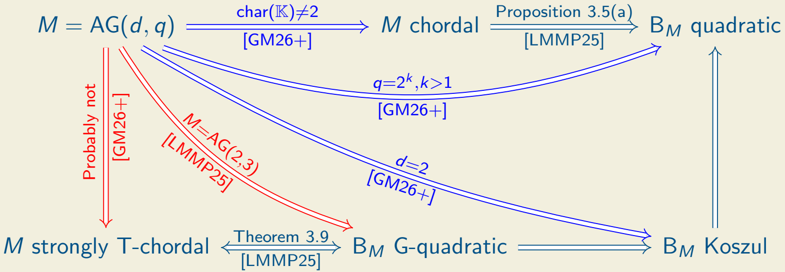
M strongly T-chordal $\xLeftrightarrow[\text{[LMMP25]}]{\text{Theorem 3.9}} \mathrm{B}_M$ G-quadratic $\Longrightarrow \mathrm{B}_M$ Koszul

B_M Koszul $\Longrightarrow \mathrm{B}_M$ quadratic

Affine Geometries

Fact

Affine geometries are obtained from projective geometries by deleting the points on the hyperplane at infinity.



Future Directions

- Can we (combinatorially) compute a graded minimal free resolution of B_M?
- What other matroids have Koszul graded Möbius algebras?
- Lattice-theoretic criterion for Koszulness of B_M?

Thank you!