Betti tables forcing failure of the weak Lefschetz property

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January 26, 2024

Artinian Algebras

Definition

A $\mathbb{K}$-algebra A is Artinian if $A_d = 0$ for $d \gg 0$.

Example: $A = \mathbb{K}[x, y, z]/ < x^2, y^2, z^3 >$

d	0	1	2	3	4	5
gens(A_d)	1	x y z	xy xz yz z^2	xyz x^2z y^2z	xyz^2	$\emptyset$
dim(A_d)	1	3	4	3	1	0

Example: If $I_\Delta \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a Stanley-Reisner ideal, then $I_\Delta + < x_i^2 \,|\, 1 \leq i \leq n >$ is Artinian.

Weak Lefschetz Property (WLP)

If A is Artinian, everything is a zero-divisor, so the best we can hope for is that there is some $f \in A$ such that $\cdot f$ is either injective or surjective in all degrees.

Definition

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and let $A = R/I$ be a standard-graded Artinian $\mathbb{K}$-algebra, where $I \subseteq R$ is a homogeneous ideal. Then A has the weak Lefschetz property (WLP) if multiplication by a general linear form

$$A_i \xrightarrow{\cdot l} A_{i+1}$$

has full rank for all $i \geq 0$, i.e., it is either injective or surjective.

Who cares?

- Geometric combinatorics
  - g-theorem (formerly McMullen's conjecture)
  - log-concavity of sequences (e.g., f-vectors)
  - top-heavy theorem
- Commutative algebra
  - Fröberg's conjecture

WLP is hard

Theorem (Harbourne, Schenck, Seceleanu)

Let

$$I = \langle L_1^t, \ldots, L_n^t \rangle \subset k[x_1, x_2, x_3, x_4]$$

with $L_i \in R_1$ *generic. If* $n \in \{5, 6, 7, 8\}$, *then the WLP fails, respectively, for* $t \geq \{3, 27, 140, 704\}$.

Theorem (Li, Zanello)

For any given positive integers a, b, c, *the number of plane partitions contained inside an* $a \times b \times c$ *box is divisible by a prime* p *if and only if* $\mathbb{K}[x, y, z]/\langle x^{a+b}, y^{a+c}, z^{b+c} \rangle$ *fails to have the WLP when* $char(\mathbb{K}) = p$.

Betti tables

Given an ideal $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$, a free resolution "approximates" R/I. The betti table records the ranks of the summands appearing in a free resolution.

Example: $R = k[x, y, z]$ and $I = < x^2, y^2, z^3 >$

$$0 \leftarrow R/I \leftarrow R \leftarrow \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \leftarrow \begin{matrix} R(-4) \\ \oplus \\ R(-5)^2 \end{matrix} \leftarrow R(-7) \leftarrow 0$$

	0	1	2	3
0	1	.	.	.
1	.	2	.	.
2	.	1	1	.
3	.	.	2	.
4	.	.	.	1

Koszul tails

Definition

A Betti table B has an (n,d)-Koszul tail if it has an upper-left principal block of the form

$$\begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ \hline 0 & 1 & . & . & . & \dots & . & . & . \\ 1 & . & . & . & . & \dots & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d-1 & . & . & . & . & \dots & . & . & . \\ d & . & n & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} & n & 1 \end{array}.$$

If B has an (n,d)-Koszul tail and is the Betti table for an Artinian ring $\mathbb{K}[x_1,\dots,x_n]/I$, then we say B has a maximal (n,d)-Koszul tail.

Koszul tails

Example:

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	.	.	.	.
3	.	3	3	1	.
4	.	44	111	90	20
5	.	.	.	.	3

$(3, 3)$-Koszul tail

	0	1	2	3
0	1	.	.	.
1	.	.	.	.
2	.	.	.	.
3	.	3	3	1
4	.	15	27	12

Maximal $(3, 3)$-Koszul tail

Results

Theorem (G., Schenck)

An Artinian algebra $A = \mathbb{K}[x_1, \ldots, x_n]/I$ whose Betti table has a maximal (n, d)-Koszul tail does not have the WLP.

Example:

	0	1	2	3
0	1	.	.	.
1	.	.	.	.
2	.	.	.	.
3	.	3	3	1
4	.	15	27	12

Results

Corollary

If $T = \mathbb{K}[x_1, \ldots, x_n]/I$ is Cohen-Macaulay of dimension m, and the Betti table of T has a maximal $(n-m, d)$-Koszul tail, then the Artinian reduction of T does not have the WLP.

Corollary

If $A = \mathbb{K}[x_1, \ldots, x_{m+n}]/I$ is Artinian with an (n, d)-Koszul tail, and there exists a sequence of linearly independent linear forms $\{l_1, \ldots, l_m\}$ such that the Betti tables of A and A/I_L have the same top row, then A/I_L does not have the WLP.

Future Work

- Can we do better than a *Koszul* tail? Having a Koszul tail is very strict.
- Is there a "nice" Boij-Söderberg theory underlying this?
- Characterize the Stanley-Reisner rings whose Artinian reductions have a Koszul tail.