Castelnuovo-Mumford Regularity of Toric Surfaces

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Castelnuovo-Mumford regularity

Definition

Let $I \subseteq \mathbb{K}[x_0, \ldots, x_n]$ be a homogeneous ideal, and consider the graded minimal free resolution

$$0 \leftarrow R/I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n+1} \leftarrow 0$$

of R/I, where $F_i \cong \bigoplus_j R(-i-j)^{\beta_{i,i+j}}$. The Castelnuovo-Mumford regularity (or simply regularity) is

$$\operatorname{reg}(R/I) = \max_{i,j} \{ j \ : \ \beta_{i,i+j} \neq 0 \} .$$

The regularity is the index of the bottom row of the Betti table.

Castelnuovo-Mumford regularity

Definition

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular if

$$H^i(\mathcal{F}(m-i)) = 0$$

for all $i > 0$. The Castelnuovo-Mumford regularity (or simply regularity) is

$$\inf\left\{d \ : \ H^i(\mathcal{F}(d-i)) = 0 \text{ for all } i > 0\right\}.$$

Monomial curves

Definition

The monomial curve with exponents $a_1 \leq \ldots \leq a_{n-1}$ in $\mathbb{P}^n$ is the curve $C \subset \mathbb{P}^n$ of degree $d = a_n$ parameterized by

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^n \quad \text{with} \quad (s,t) \mapsto (s^d, s^{d-a_1}t^{a_1}, \ldots, s^{d-a_{n-1}}t^{a_{n-1}}, t^d).$$

Theorem (L'vovsky; 1996)

Let $A = (0, a_1, \ldots, a_n)$ be a sequence of non-negative integers such that the g.c.d. of the a_j's equals 1, and let C be the corresponding monomial curve. Then C is m-regular, where

$$m = \max_{1 \leq i < j \leq n} \{(a_i - a_{i-1}) + (a_j - a_{j-1})\},$$

i.e., m is the sum of the two largest gaps in the semigroup generated by A.

Example (sporadic)

$$A = \begin{pmatrix} 0 & 3 & 5 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 3 & 5 & 7 \\ 7 & 4 & 2 & 0 \end{pmatrix}$$

$$\varphi(s,t) = (\underset{x_0}{s^7}, \quad \underset{x_1}{s^3 t^4}, \quad \underset{x_2}{s^5 t^2}, \quad \underset{x_3}{t^7})$$

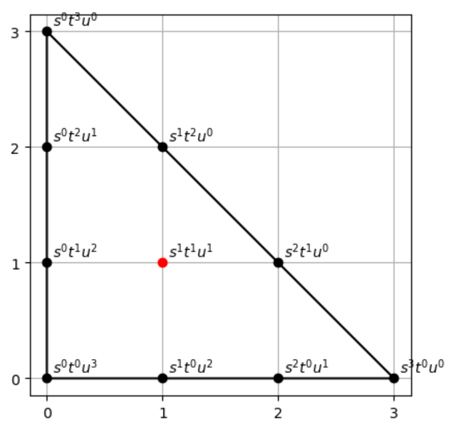
$$I_C = \langle x_2^2 - x_1 x_3, \quad x_1^3 x_2 - x_0^2 x_3^2, \quad x_1^4 - x_0^2 x_2 x_3 \rangle$$

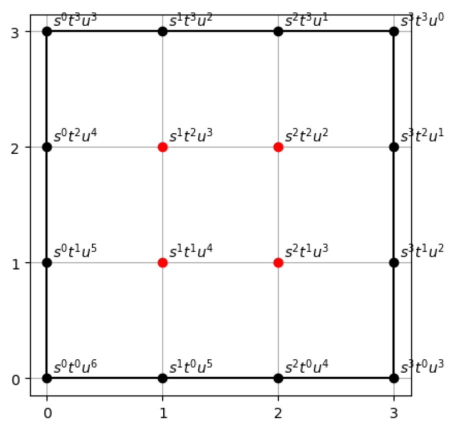
$$0 \leftarrow R/I_C \leftarrow R \leftarrow \begin{matrix} R(-2) \\ \oplus \\ R(-4)^2 \end{matrix} \leftarrow R(-5)^2 \leftarrow 0$$

$$\mathrm{reg}(I_C) = 4 \leq (a_1 - a_0) + (a_2 - a_1) = 3 + 2 = 5$$

```
i1: kk = ZZ/32749;

i2: I = monomialCurveIdeal(kk[x_0..x_3], {3,5,7})

             2          3      2 2   4    2
o2 = ideal (x  - x x , x x  - x x , x  - x x x )
             2    1 3   1 2    0 3   1    0 2 3

i3 : print betti res I
      0 1 2
total: 1 3 2
    0: 1 . .
    1: . 1 .
    2: . . .
    3: . 2 2
```

Toric surfaces





Eisenbud-Goto

Definition

A polytope P is k-normal if the map

$$\underbrace{P + P + \ldots + P}_{k \text{ times}} \longrightarrow kP$$

is surjective. Define k_P to be the smallest k such that P is k-normal.

Conjecture (Eisenbud-Goto, 1984)

For a smooth projective variety X,

$$\operatorname{reg}(X) \leq \deg(X) - codim(X) + 1.$$

In particular, for a projective toric variety coming from a polytope P,

$$k_P \leq Vol(P) - |P| + \dim P - 1.$$

What has been done?

Theorem (Lazarsfeld, 1997)

Every smooth, projective surface satisfies the Eisenbud-Goto conjecture.

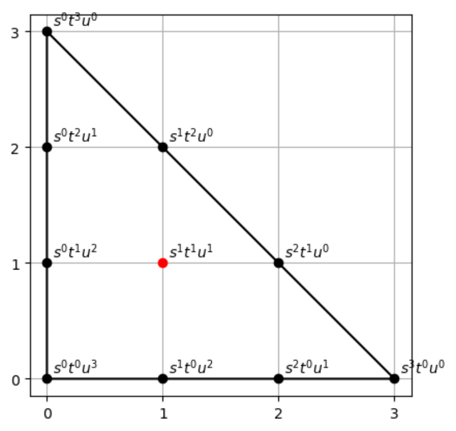
Theorem (Koelman, 1993)

For a lattice polygon P, the ideal I_P is generated by quadric and cubic binomials. Moreover, all of the minimal generators of I_P are quadrics if and only if $|\partial P| > 3$.

Theorem (Schenck, 2004; Hering, 2006)

If P has nonempty interior, then the index where $\beta_{i,i+2}$ is first nonzero is $|\partial P|$.

Setup



$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

$$\downarrow \quad \mathrm{Conv}(A) \setminus \mathrm{Int}(A)$$

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

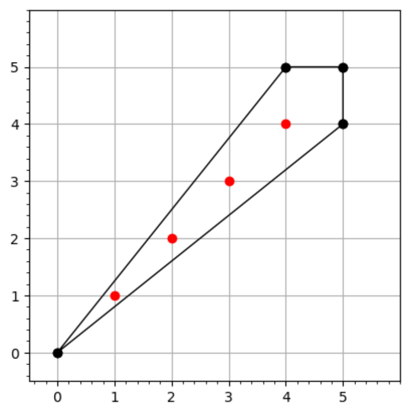
$$\downarrow \quad \text{homogenize}$$

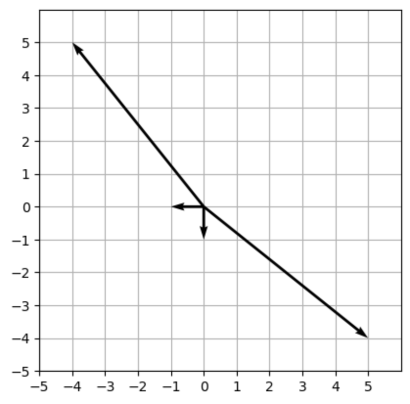
$$\tilde{A} = \begin{pmatrix} 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 2 & 1 \\ 3 & 2 & 1 & \cdots & 0 & 1 & 2 \end{pmatrix}$$

“Bad Boy”

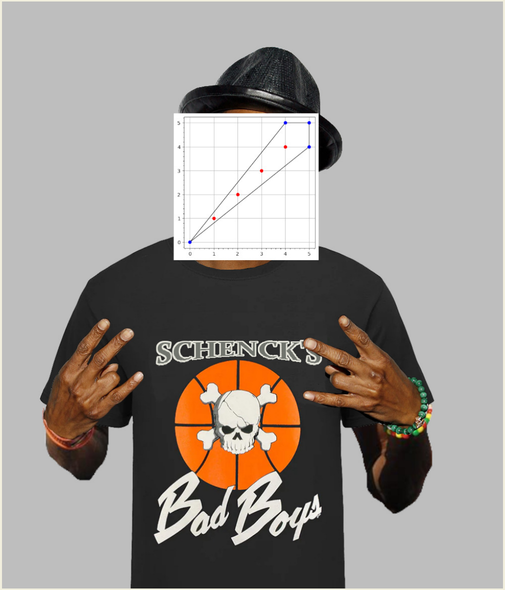
In general, the regularity can be arbitrarily large by using

$$A = \begin{pmatrix} 0 & d & d-1 & d \\ 0 & d-1 & d & d \end{pmatrix}.$$





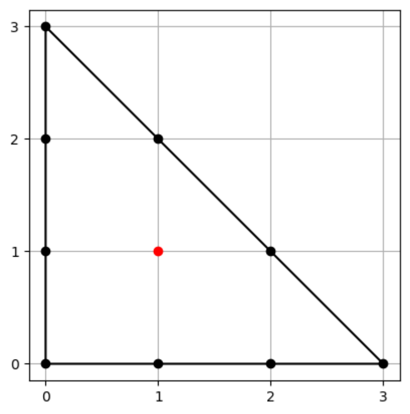
“Bad Boy”

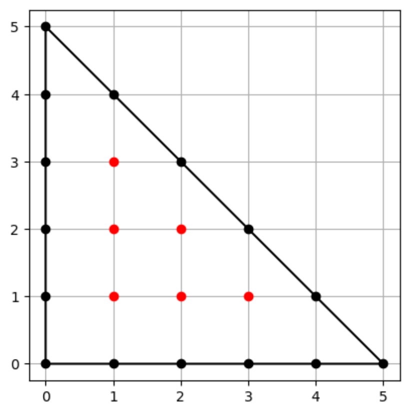


Hollow triangle

Definition

Suppose $A = \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$. *The hollow triangle of length* k *is* $\triangle^k := \tilde{A}$.





Hollow triangle data

```
+---+-----------------------------------+
|   |                                   |
|   |          0 1 2 3                  |
| 2 | total: 1 6 8 3                    |
|   |     0: 1 . . .                    |
|   |     1: . 6 8 3                    |
|   |                                   |
+---+-----------------------------------+
|   |                                   |
|   |          0  1  2  3   4  5  6 7 8 |
| 3 | total: 1 17 53 91 108 83 37 9 1 |
|   |     0: 1  .  .  .   .  .  . . . |
|   |     1: . 17 43 36   8  .  . . . |
|   |     2: .  . 10 55 100 83 37 9 1 |
|   |                                   |
+---+-----------------------------------+
```

Hollow triangle data

```
+---+------------------------------------------------------
|   |
|   |          0  1   2   3    4    5    6    7   8   9 10 11
| 4 | total: 1 33 153 525 1356 2178 2205 1486 675 201 36  3
|   |     0: 1  .   .   .    .    .    .    .   .   .  .  .
|   |     1: . 33 123 144   30    .    .    .   .   .  .  .
|   |     2: .  .  30 381 1326 2178 2205 1486 675 201 36  3
|   |
+---+------------------------------------------------------
|   |
|   |          0  1   2    3    4     5     6     7     8     9
| 5 | total: 1 54 389 2028 7845 18957 30393 34672 29106 18162
|   |     0: 1  .   .    .    .     .     .     .     .     .
|   |     1: . 54 266  462  174    15     .     .     .     .
|   |     2: .  . 123 1566 7671 18942 30393 34672 29106 18162
|   |
+---+------------------------------------------------------
```

Results

Lemma

For all $d \geq 2$, $(R/I_{\triangle^k})_d = (\overline{R/I_{\triangle^k}})_d$.

Theorem

For all $k \geq 2$, $\operatorname{reg}(\triangle^k) = 2$.

Results

Lemma

For all $d \geq 2$, $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$.

Theorem

For all $k \geq 2$, $\operatorname{reg}(\square^k) = 2$.

Proof sketch of theorem

- *Use the short exact sequence of sheaves*

$$0 \to \mathscr{I}_{\square^k}(d) \to \mathscr{O}_{\mathbb{P}^{4k-1}}(d) \to \mathscr{O}_{\square^k}(d) \to 0$$

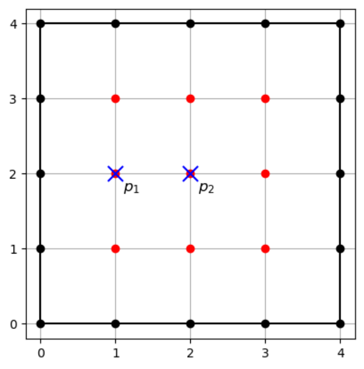
  to eventually get a short exact sequence

$$0 \to R/I_{\square^k} \to \overline{R/I_{\square^k}} \to N \to 0.$$

- *By the lemma, N is generated is only generated by degree* 1 *monomials.*
- $\mathrm{reg}(R/I_{\square^k}) \leq \max(\mathrm{reg}(\overline{R/I_{\square^k}}), N) = \mathrm{reg}(\overline{R/I_{\square^k}}) = 2.$

Proof sketch of lemma

- *Showing* $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$ *for* $d \geq 2$ *amounts to a computation with the lattice points of* $\square^k$.
- *We are done if for any* $p_1, p_2 \in \overline{\square^k}$, *we can write* $p_1 + p_2 = q_1 + q_2$ *with* $q_1, q_2 \in \square^k$.



$$p_1 + p_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$$

Proof sketch of lemma

- *Showing* $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$ *for* $d \geq 2$ *amounts to a computation with the lattice points of* $\square^k$.
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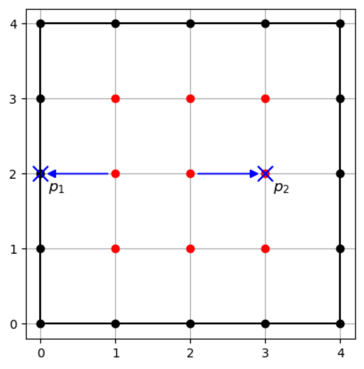

$$p_1 + p_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Proof sketch of lemma

- *Showing* $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$ *for* $d \geq 2$ *amounts to a computation with the lattice points of* $\square^k$.
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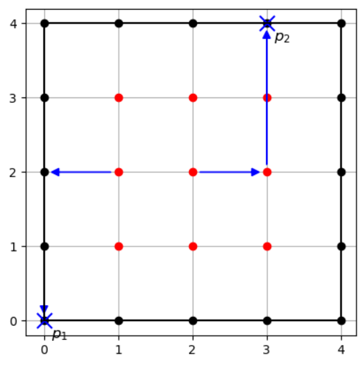
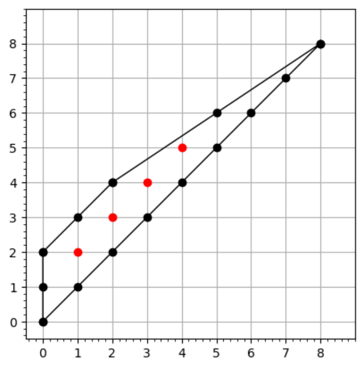
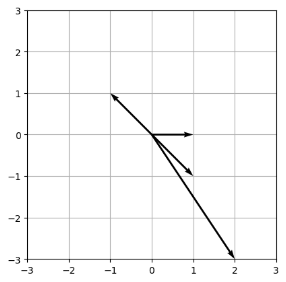


$$p_1 + p_2 = \begin{pmatrix}1\\2\\5\end{pmatrix} + \begin{pmatrix}2\\2\\4\end{pmatrix} = \begin{pmatrix}3\\4\\9\end{pmatrix}$$
$$= \begin{pmatrix}0\\2\\6\end{pmatrix} + \begin{pmatrix}3\\2\\3\end{pmatrix}$$
$$= \begin{pmatrix}0\\0\\8\end{pmatrix} + \begin{pmatrix}3\\4\\1\end{pmatrix}$$

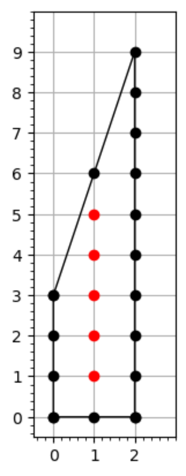
Smooth is not enough

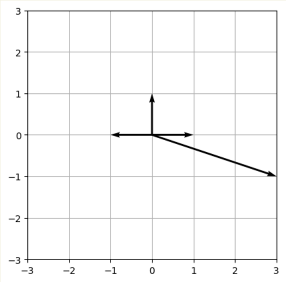




	0	1	2	3	4	5	...
total:	1	54	385	1462	3608	6456	...
0:	1	.	.	.	.	.	...
1:	.	52	280	730	1128	1050	...
2:	.	.	81	600	2040	4416	...
3:	.	2	24	132	440	990	...

Smooth is not enough





	0	1	2	3	4	5	...
total:	1	74	633	2883	8593	18953	...
0:	1	.	.	.	.	.	...
1:	.	73	486	1627	3388	4620	...
2:	.	1	144	1218	4983	13541	...
3:	.	.	3	38	222	792	...

Analog of Gaps

Conjecture

Suppose X is a smooth toric surface in the setup of the hollow polygons. Let Σ be its corresponding fan with rays $\rho_1, \ldots, \rho_r$. Then the regularity of I_X is bounded by the largest magnitude of a ray of Σ, i.e.,

$$\operatorname{reg}(R/I_X) \leq \max_{1 \leq i \leq r} \|\rho_i\|_1 .$$

Alternatively, if $v_1, \ldots, v_r = v_0$ are the vertices of the corresponding polytope P (ordered clockwise), then

$$\operatorname{reg}(R/I_X) \leq \max_{1 \leq i \leq r} \frac{\|v_i - v_{i-1}\|_1}{\gcd(v_i - v_{i-1})}.$$