# Homological Properties of the Graded Möbius Algebra of a Matroid

Commutative Algebra Seminar



**Sean Grate** · Iowa State University

Joint with Jason McCullough, and with Louiza Fouli, Selvi Kara, Adam LaClair, Jason McCullough, Vinh Nguyen, Aleksandra Sobieska, Prajwal Udanshive

# Macaulay2 at Auburn 2027



**February 22-25th, 2027**

Organizers: John Cobb, Sean Grate, Michael Brown

https://johndcobb.github.io/organizing/M2atAuburn/

**Objectives**

- Develop new software for Macaulay2 and its packages
- Form new collaborations between different mathematical areas
- Train students through hands-on learning and coding

| Group Leader(s) | Focus |
|---|---|
| Keller VandeBogert | *Varieties* package |
| Jay Yang | resolutions of toric subvarieties |
| Thomas Brazelton | Grothendieck–Witt rings |
| Jordy Garcia-Lopez | Tropical intersections |
| Sean Grate | matroids |
| Doug Torrance | $p$-adic numbers |
| Juliette Bruce and John Cobb | toric stacks |

# The Crew



Sean Grate (Me)




Jason McCullough

# Matroids

A matroid abstracts the notion of linear independence in vector spaces.

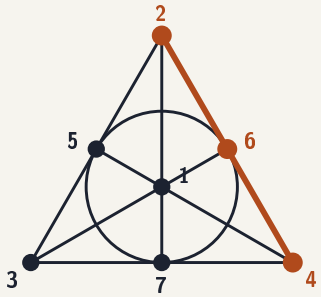
$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

**rank** $\operatorname{rk} T$ = rank of the columns in $T$

**basis** maximal independent set: 3 columns spanning $\mathbb{F}_2^3$ (there are 28)

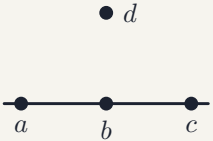
**flat** closed under span: $\{2, 4, 6\} = \operatorname{span}(4, 6)$

**circuit** minimal dependent set: $\operatorname{col}_2 = \operatorname{col}_4 + \operatorname{col}_6$




The Fano plane $\mathrm{PG}(2, 2)$. Its 7 lines are the 2-dimensional subspaces.

# The lattice of flats

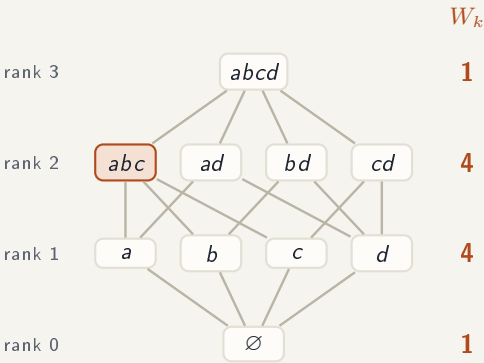



$a, b, c$ on a line, $d$ off it

Flats ordered by inclusion: join = span, meet = intersection, graded by rank.

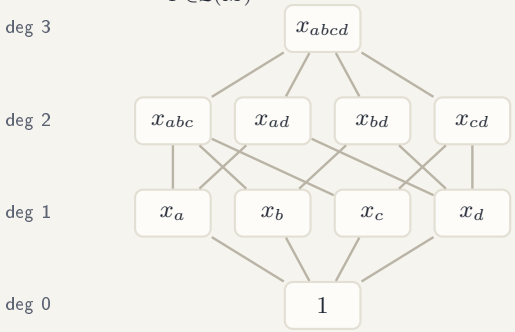
The **Whitney numbers (of the second kind)** count flats of each rank:

$$W_k = \#\{F : \operatorname{rk} F = k\}$$



# The graded Möbius algebra

$$\mathrm{GMA}(M) = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K}\, x_F, \qquad x_F \cdot x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G), \\ 0 & \text{otherwise.} \end{cases}$$

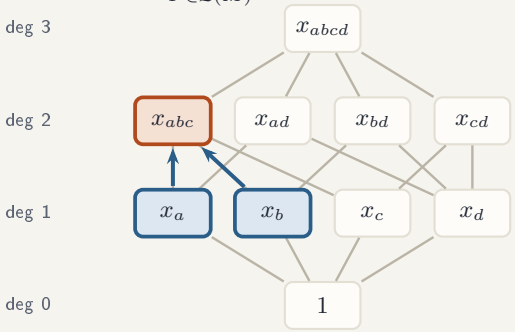



**Basis = the flats**, degree = rank.

Hilbert function $(1, 4, 4, 1)$: the Whitney numbers.

# The graded Möbius algebra

$$\mathrm{GMA}(M) = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K}\, x_F, \qquad x_F \cdot x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G), \\ 0 & \text{otherwise.} \end{cases}$$

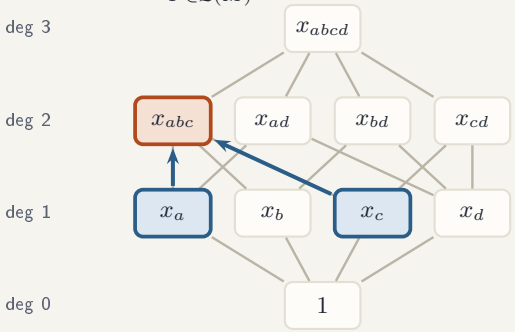



$$x_a \cdot x_b = x_{abc}$$

The join is $abc$, and the rank adds: $1 + 1 = 2$.

# The graded Möbius algebra

$$\mathrm{GMA}(M) = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K}\, x_F, \qquad x_F \cdot x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G), \\ 0 & \text{otherwise.} \end{cases}$$

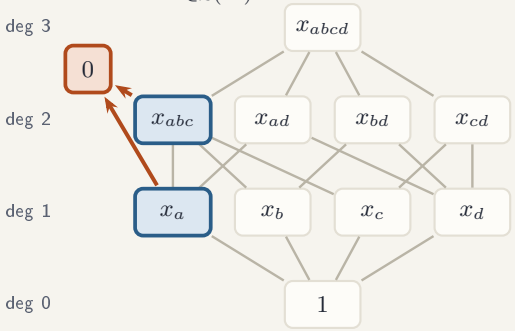



$$x_a \cdot x_c = x_{abc}$$

Same join, so $x_a x_b = x_a x_c$ yielding the relation $ab - ac$.

# The graded Möbius algebra

$$\mathrm{GMA}(M) = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K}\, x_F, \qquad x_F \cdot x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G), \\ 0 & \text{otherwise.} \end{cases}$$

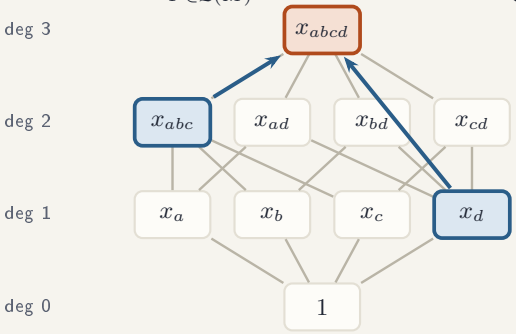



$$x_a \cdot x_{abc} = 0$$

The join is $abc$ again, but the rank does not add: $1 + 2 \not< 2$.

# The graded Möbius algebra

$$\mathrm{GMA}(M) = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K}\, x_F, \qquad x_F \cdot x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G), \\ 0 & \text{otherwise.} \end{cases}$$




$$x_d \cdot x_{abc} = x_{abcd}$$

The rank adds, $1 + 2 = 3$, and we reach the top of the lattice.

# The Whitney numbers are top-heavy

**Theorem**

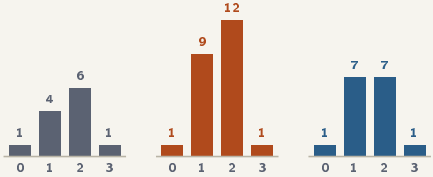
Greene (1970)

$W_1 \leq W_{r-1}$, with equality iff the lattice of flats is **modular**.

**Top-heavy theorem (Dowling–Wilson conjecture, 1974)**

Huh–Wang (2017)
Braden–Huh–Matherne–Proudfoot–Wang (2026)

$$W_k \leq W_j \quad \text{whenever } k \leq j \leq r-k.$$

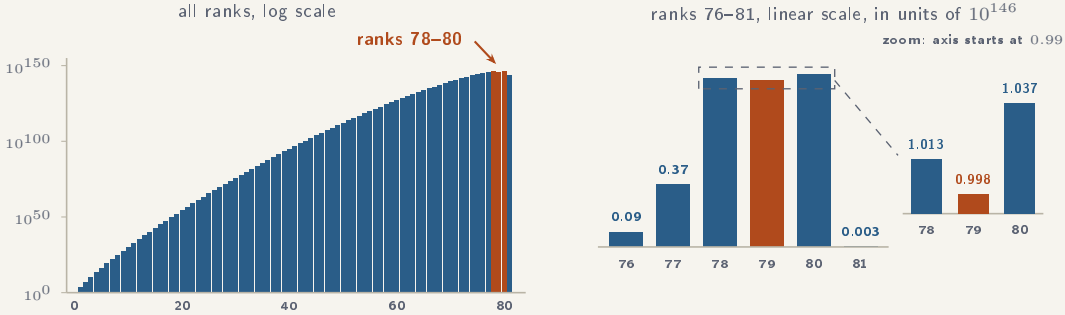



Two are strictly top-heavy; the Fano plane is symmetric.

$\mathrm{GMA}(M)$ sits inside the intersection cohomology module $IH(M)$, which has hard Lefschetz. So, multiplication by a power of a linear form injects $\mathrm{GMA}(M)_k \hookrightarrow \mathrm{GMA}(M)_{r-k}$.

# Not unimodal

Rota conjectured $(W_0, \ldots, W_r)$ is **unimodal** (1970) and Mason strengthened the conjecture to **log-concave** (1971). Top-heavy gives $W_0 \leq \cdots \leq W_{\lfloor r/2 \rfloor}$.



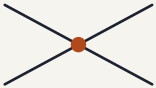

The 59-lift of the theta graph with paths of lengths 1, 27, 28, 28: rank 82, 4957 elements, **peaks at ranks 78 and 80** (Divoux–Larson–Lowen–Wang, 2026).

# Gorenstein = symmetric = projective

**Theorem (Greene; Maeno–Numata)**

For a simple matroid $M$, the following are equivalent:

1. $\mathrm{GMA}(M)$ is **Gorenstein**;
2. its Hilbert function $(W_0, \ldots, W_r)$ is **symmetric**;
3. the lattice of flats is **modular**: $\operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G) + \operatorname{rk}(F \wedge G)$.



$2 + 2 = 3 + 1$

projective: lines meet



$2 + 2 \neq 3 + 0$

affine: parallel lines

By Birkhoff, modular means a product of **projective geometries** (with Boolean and rank-2 pieces).

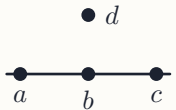
Then $\mathrm{GMA}(M) = IH(M)$: Gorenstein with hard Lefschetz and Hodge–Riemann.

# Presentation of $\mathrm{GMA}(M)$

$$\mathrm{GMA}(M) \;\cong\; \frac{\mathbb{K}[y_i : i \in E]}{\big(y_i^2 : i \in E\big) + \big(y_{C\setminus i} - y_{C\setminus j} : C \text{ a circuit},\ i,j \in C\big)}$$

(LaClair–Mastroeni–McCullough–Peeva, 2025)

A binomial ideal containing the squares. **A circuit with $c$ points gives relations of degree $c-1$.**




$I = (a^2, b^2, c^2, d^2,\ ab - ac,\ ab - bc)$

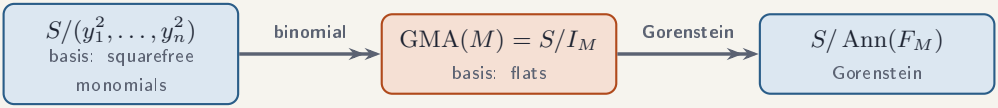
**quadratic · $h = (1,4,4,1)$ · Gorenstein**




$I = (a^2, b^2, c^2, d^2,\ abc{-}abd,\ abc{-}acd,\ abc{-}bcd)$

**cubic relations · $h = (1,4,6,1)$ · not Gorenstein**

# Algebraic translation




The first map identifies independent sets with the same span and kills dependent ones. The second kills everything that pairs to $0$ into the top degree; by Macaulay duality it is $S/\operatorname{Ann}(F_M)$, where $F_M = \sum_{B \text{ basis}} y_B$ is the **basis generating polynomial**.

**So, in this language**

$I_M \subseteq \operatorname{Ann}(F_M)$ always, with equality $\iff$ $\mathrm{GMA}(M)$ is Gorenstein $\iff$ $M$ is modular.

| | $h(\mathrm{GMA})$ | $h(S/\operatorname{Ann} F_M)$ |
|---|---|---|
| Fano plane | $(1,7,7,1)$ | $(1,7,7,1)$ |
| 4 general points | $(1,4,6,1)$ | $(1,4,4,1)$ |
| $\mathrm{AG}(2,3)$ | $(1,9,12,1)$ | $(1,9,9,1)$ |

# When is $\mathrm{GMA}(M)$ Koszul?

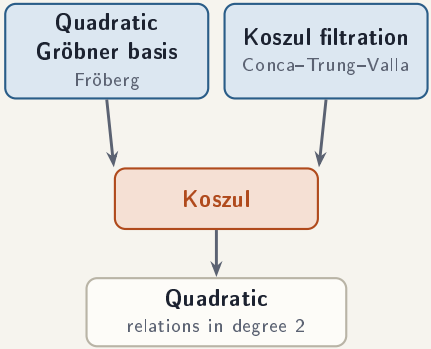
A standard graded $\mathbb{K}$-algebra $A = S/I$ is **Koszul** if $\mathbb{K}$ has a **linear** free resolution over $A$,

$$\cdots \longrightarrow A(-2)^{\beta_2} \longrightarrow A(-1)^{\beta_1} \longrightarrow A \longrightarrow \mathbb{K} \longrightarrow 0,$$

and then $P_{\mathbb{K}}^{A}(t) \cdot H_A(-t) = 1$.

There is no finite test (Roos; Gorenstein examples by McCullough–Seceleanu).

Not every $\mathrm{GMA}(M)$ is Koszul: four general points already give cubic relations. **Which matroids, and by which route?**



# Parallels with Orlik–Solomon algebras

| | Orlik–Solomon algebra | Graded Möbius algebra |
|---|---|---|
| Quadratic Gröbner basis $\Leftrightarrow$ | $M$ supersolvable | $M$ strongly T-chordal |
| Graphic $M(G)$: Koszul $\Leftrightarrow$ | $G$ chordal | $G$ strongly chordal |
| Graphic: quadratic $\Rightarrow$ Koszul? | yes | no |
| Koszul $\Rightarrow$ quadratic GB? | no (Le–Lowen–McCullough) | no (e.g. $\mathrm{AG}(2,3)$) |

**Quadratic** (LaClair–Mastroeni–McCullough–Peeva, 2025)

If the matroid is chordal, i.e. every circuit of size $\geq 4$ splits along a chord, then $\mathrm{GMA}(M)$ is quadratic.

THEOREM · WITH J. MCCULLOUGH

# If $\mathrm{GMA}(M)$ is Gorenstein, then $\mathrm{GMA}(M)$ is Koszul.

Equivalently: for every modular matroid $M$, the algebra $S/\operatorname{Ann}(F_M)$, where $F_M$ is the matroid's basis generating polynomial, has a **quadratic Gröbner basis**, hence is Koszul.

**Not automatic.**

For all $c \geq r + 2 \geq 6$ there are quadratic Gorenstein algebras of regularity $r$ and codimension $c$ that are not Koszul (Mastroeni–Schenck–Stillman).

# Reducing to projective geometries

**Birkhoff (1935)**

A finite modular geometric lattice is a product of:

- Boolean lattices
- rank-2 lattices
- projective planes
- subspace lattices $\mathrm{PG}(n, q)$

$$\mathrm{GMA}(M \oplus N) \;\cong\; \mathrm{GMA}(M) \otimes_k \mathrm{GMA}(N)$$

Quadratic Gröbner bases survive $\otimes$, the Boolean and rank-2 pieces are easy, so it suffices to treat **projective planes and** $\mathrm{PG}(n, q)$.

Quadratic is the easy part: a line always meets a hyperplane, so every circuit has a "chord" and its relation reduces to quadrics.

# Quadratic Gröbner bases: the LMMP criterion

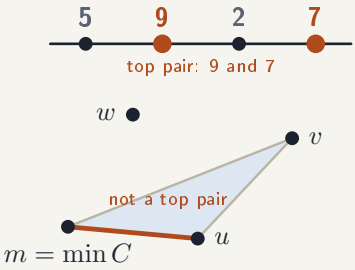
**Theorem (LaClair–Mastroeni–McCullough–Peeva)**

If $\mathrm{GMA}(M)$ is quadratic, it has a quadratic Gröbner basis in its natural presentation **iff** some total order $\succ$ on $E$ makes every 4-circuit an **MAT-circuit**.

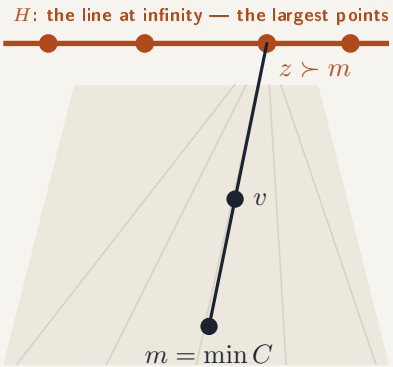
**Top pair** of a line: its two $\succ$-largest points.

**MAT-circuit**: a 4-circuit $C$ such that for every $i \neq \min C$, the triangle $C \setminus i$ has a side that is **not** a top pair.

Top pairs are the standard quadrics, and three of the four cubic monomials of $C$ must be non-standard.



# Proof idea: points at infinity go on top

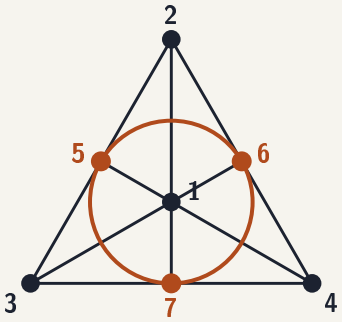



Order the points so that a line $H$ "at infinity" comes **last**.

Then for a 4-circuit with minimum $m$, the line from $m$ to another affine point $v$ hits $H$ at a larger point $z$, and so $\{m, v\}$ is not a top pair.

After some special cases, induct on dimension.

# Example




Order $1 \prec \cdots \prec 7$, so the line at infinity $\{x_1 = 0\} = \{5, 6, 7\}$ comes last. Grevlex with $y_1 > \cdots > y_7$:

$$\operatorname{in}(I_M) = \left(y_1^2, \ldots, y_7^2\right) + \left(14 \text{ squarefree quadrics}\right)$$

Standard monomials:

| | |
|---|---|
| degree 0 | $1$ |
| degree 1 | $y_1,\ \ldots,\ y_7$ |
| degree 2 | $y_2y_7,\ \ y_3y_5,\ \ y_3y_6,\ \ y_4y_5,\ \ y_4y_6,\ \ y_4y_7,\ \ y_6y_7$ |
| degree 3 | $y_4y_6y_7$ |

Degree 2: exactly the **top pair of each line**.

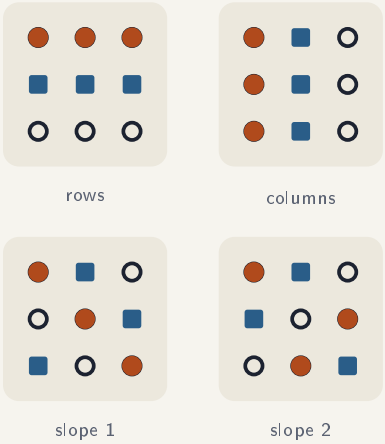
# Affine geometries

**$\mathbf{AG}(n, q)$**: delete a hyperplane from $\mathrm{PG}(n, q)$.

Parallel lines $\Rightarrow$ **not modular** $\Rightarrow$ never Gorenstein ($n \geq 2$).

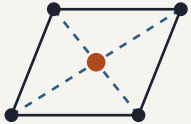
$$W_k\big(\mathrm{AG}(n,q)\big) = q^{\,n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \qquad (k \geq 1)$$

$\mathrm{AG}(2,3)$: 9 points, 12 lines in 4 parallel classes; $h = (1, 9, 12, 1)$.

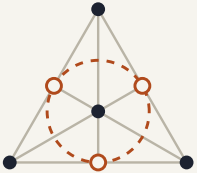


# Which affine geometries are quadratic?

A circuit relation reduces to quadrics when two opposite sides of the circuit meet in a point.



$q$ odd: the diagonals meet



$q = 2$: the meeting points are all at infinity

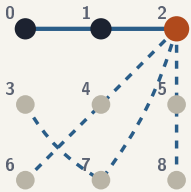
| | |
|---|---|
| $q$ **odd** | quadratic |
| $q = 2$ | lines have 2 points: **not** quadratic |
| $q \geq 4$ **even** | still quadratic, by a different route |

**Theorem** (with J. McCullough). For $n \geq 2$, $\mathrm{GMA}(\mathrm{AG}(n, q))$ is quadratic if and only if $q \geq 3$.

# $\mathrm{AG}(2,3)$: Koszul without a quadratic Gröbner basis

**Theorem (with J. McCullough)**

For $n \geq 2$, no monomial order gives $I_{\mathrm{AG}(n,3)}$ a quadratic Gröbner basis in the variables $y_i$.




**Theorem (with J. McCullough)**

$\mathrm{GMA}(\mathrm{AG}(2,3))$ is Koszul.

The proof is a **Koszul filtration**, and its colon ideals can be read off the lines. Adding point $i$ to earlier points, each line through $i$ contributes:

- solid, meets an earlier point: its points;
- dashed, misses them: the difference of its other two points.

$$(y_0, y_1) : y_2 = (y_0, y_1, y_2,\ \ y_3 - y_7,\ y_4 - y_6,\ y_5 - y_8)$$

# Evidence for Koszulness

**Theorem (with J. McCullough)**

For $\mathrm{PG}(n,q)$, and for $\mathrm{AG}(n,q)$ with $q \geq 3$, the Hilbert series of the GMA is **real-rooted**. So $1/H(-t)$ has nonnegative coefficients.

Proof: Hutchinson's criterion $a_i^2 > 4\,a_{i-1}a_{i+1}$, checked on $q$-binomial coefficients.

$\mathrm{AG}(2,2)$: **not quadratic**

$$\frac{1}{H(-t)} = 1+4t+10t^2+17t^3+12t^4-44t^5-\cdots$$

$\mathrm{AG}(2,3)$: **Koszul**

$$\frac{1}{H(-t)} = 1+9t+69t^2+514t^3+3807t^4+\cdots$$

# Beyond Koszul: resolutions over $S$

$S/I_M$ is Artinian, so $\mathrm{pd} = n$ and $\mathrm{reg} = r$. What is the whole Betti table?

For $n$ points in general position in rank $r$—the **uniform matroid** $U_{r,n}$—the algebra is easy to describe:

$S/I_M$ is $S/(y_1^2, \ldots, y_n^2)$ with all squarefree monomials of degree $r$ **identified** with one another (and degree $r+1$ killed).

Not quadratic once $r \geq 3$, so not Koszul, but the whole resolution is explicit.

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0: | 1 | · | · | · | · |
| 1: | · | 4 | · | · | · |
| 2: | · | 3 | 17 | 14 | 2 |
| 3: | · | · | · | · | 1 |

Betti table of $S/I_M$ for $U_{3,4}$: four general points in a plane



# Linkage with the squares

1 Link through the complete intersection of squares:

$$D = (y_1^2, \dots, y_n^2) \subseteq I_M, \quad J := D : I_M, \qquad 0 \to \omega_{S/J}(-n) \to S/D \to S/I_M \to 0.$$

**2 Proposition**

With $d = n - r$, $\ell_d = \sum_{|T|=d} y_T$, and $\mathfrak{m}^{[d+1]}$ the squarefree monomials of degree $d+1$:

$$J = D + \mathfrak{m}^{[d+1]} + (\ell_d).$$

3 A mapping cone then gives the minimal resolution of $S/I_M$.

# Betti tables of uniform matroids

**Theorem**

(Fouli–G.–Kara–LaClair–McCullough–Nguyen–Sobieska–Udanshive)

For $2 \leq r < n$, the Betti table of $S/I$ for $U_{r,n}$ has exactly three parts:

- Koszul complex on the squares: $\beta_{i,2i} = \binom{n}{i}$ for $i \leq r-2$
- one row, $r-1$, with explicit alternating sums of binomial products
- the socle corner: $\beta_{n,\,n+r} = 1$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0: | 1 | | | | | | | | |
| 1: | | 8 | | | | | | | |
| 2: | | | 28 | | | | | | |
| 3: | | | | 56 | | | | | |
| 4: | | | | | 70 | | | | |
| 5: | | 27 | 208 | 693 | 1296 | 1526 | 1008 | 350 | 48 |
| 6: | | | | | | | | | 1 |

Betti table of $S/I$ for $U_{6,8}$

# When is the defining ideal linearly presented?

**Lemma**

Let $I \subseteq S$ be quadratic with $y_a^2, y_b^2 \in I$. If the monomial $y_a y_b$ appears in no element of $I_2$, then $I$ has a quadratic first syzygy: $\beta_{2,4}(S/I) \neq 0$.

Proof: $y_a y_b \otimes e_a \wedge e_b$ is a Koszul cycle, since $y_a^2 = y_b^2 = 0$, and it cannot be a boundary.

For $\mathrm{GMA}(M)$: $y_a y_b$ appears in $(I_M)_2$ exactly when $a$ and $b$ lie on a line with at least 3 points. Call $M$ **Sylvester** if this holds for every pair.

**Conjecture (Fouli–G.–Kara–LaClair–McCullough–Nguyen–Sobieska–Udanshive)**

If $I_M$ is quadratic and $|E| \geq 3$, then $I_M$ is linearly presented $\iff$ every squarefree $y_a y_b$ appears in $(I_M)_2$ $\iff$ $M$ is Sylvester.

# Examples

| Matroid | Sylvester? | $\beta_{2,4}(S/I_M)$ | Also |
|---|---|---|---|
| Fano plane $\mathrm{PG}(2,2)$ | yes | 0 | Gorenstein, Koszul |
| Affine plane $\mathrm{AG}(2,3)$ | yes | 0 | Koszul, no quadratic GB |
| One line, $n \geq 3$ points ($U_{2,n}$) | yes | 0 | Gorenstein |
| Graphic matroid of $K_4$ | no | 3 | Koszul |
| Two disjoint 3-point lines | no | 25 | Gorenstein, Koszul |

**Castelnuovo–Mumford regularity**

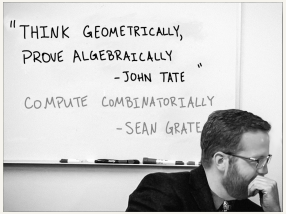
McCullough asked whether quadratic, linearly presented ideals in $n$ variables satisfy $\limsup \operatorname{reg} I_n/\sqrt{n} < \infty$. For GMAs yes: Sylvester forces $n \geq 2^r - 1$ (Murty), so $\operatorname{reg} I_M = r + 1 = O(\log n)$.

# Summary

- ✓ **Modular $M$:** the Macaulay inverse system of $F_M = \sum_B y_B$, which equals $\mathrm{GMA}(M)$, has a quadratic Gröbner basis, so it is Koszul.
- ✓ **Affine $\mathrm{AG}(n,q)$, $n \geq 2$:** quadratic $\iff$ $q \geq 3$. For $q = 3$ there is no quadratic Gröbner basis in the $y_i$, yet $\mathrm{AG}(2,3)$ is Koszul.
- ✓ **Hilbert series** of $\mathrm{GMA}(\mathrm{PG}(n,q))$ and $\mathrm{GMA}(\mathrm{AG}(n,q))$, $q \geq 3$, are real-rooted.
- ✓ **Uniform matroids:** complete Betti tables, via linkage with $(y_1^2, \ldots, y_n^2)$.
- ✓ **Linear presentation** forces every $y_a y_b$ to appear among the quadrics.

# Questions

1. **Koszul?** Is $\mathrm{GMA}(\mathrm{AG}(n,q))$ Koszul for every $q \geq 3$? (Conjectured for $n = 2$.)
2. **A better coordinate system?** $\mathrm{GMA}(\mathrm{AG}(2,3))$ is Koszul with no quadratic Gröbner basis in the $y_i$. Does it have one after a linear change of coordinates?
3. **Linear presentation.** If $I_M$ is quadratic and every squarefree $y_a y_b$ appears in $(I_M)_2$, is $I_M$ linearly presented?
4. **Syzygies of the Gorenstein ones.** What is the Betti table of $S/\operatorname{Ann}(F_M)$ for $M = \mathrm{PG}(n,q)$?





# Thank you!