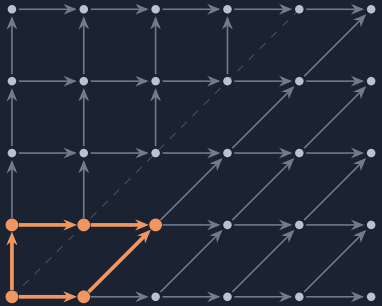
The Stable Tamari Lattice

Herman Chau · Spencer Daugherty · Sean Grate · Mark Haiman · Juan Carlos Martínez Mori · and Anna Pun

Discrete Math Seminar



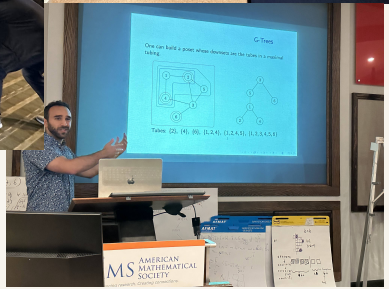
AMS MRC: Algebraic Combinatorics



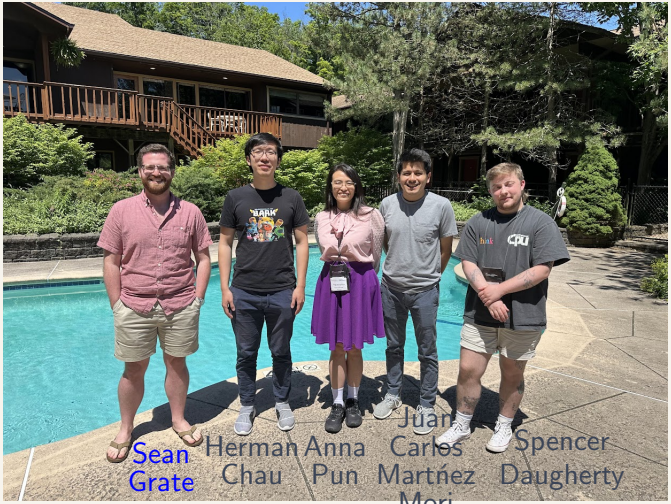








The Crew





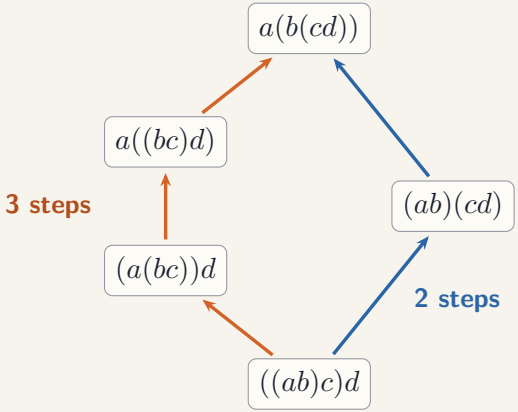
Mark Haiman

The Tamari lattice

Orient each use of the associative law

$$(xy)z \longrightarrow x(yz)$$

Its two maximal chains have lengths 3 and 2, so the Tamari lattice is **not graded**.



The **Tamari lattice**

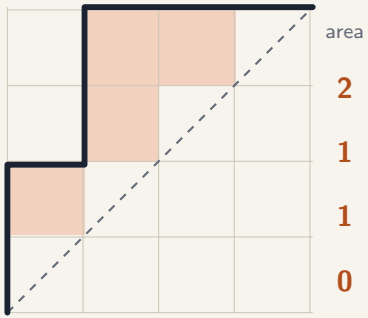
Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}: \qquad 1,\ 1,\ 2,\ 5,\ 14,\ 42,\ \ldots$$

C_n counts parenthesizations of $n+1$ factors, binary trees, triangulations, and **Dyck paths**.

A Dyck path's **area sequence** counts the full squares between path and diagonal in each row, from the top.

> Area sequences = sequences ending in 0 whose entries are at most one more than the next.



area sequence $(2, 1, 1, 0)$

Diagonal coinvariants

One set of variables (Chevalley)

S_n permutes $x_1, \ldots, x_n$. Divide by invariants with no constant term:

$$R_n = \mathbb{C}[\mathbf{x}] / \langle \mathbb{C}[\mathbf{x}]_+^{S_n} \rangle \qquad \dim R_n = n!$$

Two sets of variables (Garsia–Haiman)

S_n permutes $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ *simultaneously*:

$$DR_n = \mathbb{C}[\mathbf{x}, \mathbf{y}] / \langle \mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{S_n} \rangle$$

graded by x-degree and y-degree. ($n = 2$: basis $1,\ x_2 - x_1,\ y_2 - y_1$.)

Theorem (Haiman 2002, via the Hilbert scheme)

- $\dim DR_n = (n+1)^{n-1}$, the number of *parking functions*.
- Its sign-isotypic part has dimension C_n, with Hilbert series the q,t-Catalan polynomial.
- Its graded character is ∇e_n (Macdonald polynomials).

Shuffle theorem (Carlsson–Mellit 2018): a formula for ∇e_n over parking functions.

Trivariate diagonal coinvariants

higher, $r \geq 1$ ($r = 1$: classical)	whole space	sign-isotypic part
two sets $\mathbf{x}, \mathbf{y}$	$(rn+1)^{n-1}$ r-parking functions (theorem, Haiman)	$\frac{1}{rn+1}\binom{(r+1)n}{n}$ r-Dyck paths (theorem)
three sets $\mathbf{x}, \mathbf{y}, \mathbf{z}$	$(r+1)^n(rn+1)^{n-2}$ labelled r-Tamari intervals (dimension conjectural)	$\frac{r+1}{n(rn+1)}\binom{(r+1)^2n+r}{n-1}$ r-Tamari intervals (dimension conjectural)

For $r = 1$, three sets: $2^n(n+1)^{n-2}$ and $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ (Haiman's observations, 1994). The *counts* are theorems (Chapoton; Bousquet-Mélou–Fusy–Préville-Ratelle; Bousquet-Mélou–Chapuy–Préville-Ratelle). That they are *dimensions* is open.

Where longest chains enter

The left side is the graded character of the higher trivariate space, with the third grading set to 1.

Conjecture (Bergeron–Préville-Ratelle 2012)

$$\mathcal{H}_n^{(r)}(\mathbf{w}; q_1, q_2, 1) = \sum_{\beta} \sum_{f \in \mathrm{PF}(\beta)} i_\beta^{(r)}(q_1)\, q_2^{\mathrm{dinv}(f)}\, Q_{\mathrm{co}(f)}(\mathbf{w})$$

- β: r-Dyck paths; f: parking functions of shape β (labellings of its north steps).
- $i_\beta^{(r)}(q) = \sum_{\alpha \le \beta} q^{\,d(\alpha,\beta)}$: the **longest-chain** polynomial of the r-Tamari lattice.
- $q_1 = 0$: the shuffle formula. $q_1 = q_2 = 1$: labelled r-Tamari intervals (proved).

Still open: this conjecture, the trivariate dimensions, and a statistic for the third grading q_3, even for $r = 1$.

To attack this, one wants the polynomials $i_\beta^{(r)}$ for all r and β at once.

The plan

Goal. Understand the longest-chain polynomials $i_\beta^{(r)}(q) = \sum_{\alpha \le \beta} q^{d(\alpha,\beta)}$ for every r and every r-Dyck path β, *all at once*.

Step 1 · one home

Obstacle: a different finite lattice for each r (and each boundary ν).

Find one infinite lattice on $\mathbb{N}^n$ containing all of them.

Step 2 · measure

Obstacle: Tamari lattices are not graded, so longest chains are hard to see.

Find coordinates in which the longest chain is a sum.

Step 3 · count

Obstacle: infinitely many polynomials, one for each r and β.

Slice downsets, and group intervals by the h-vector of their bottom.

THE PLAN

1 **Step 1: one home**
a single lattice containing every r-Tamari lattice

2 Step 2: measure
coordinates in which longest chains become sums

3 Step 3: count
all the polynomials $i_\beta^{(r)}$ in one finite formula

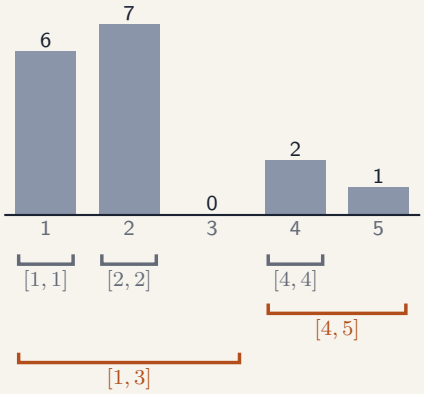
Looking left from each position

Fix $a \in \mathbb{N}^n$; here $a = (6, 7, 0, 2, 1)$.

Position j **sees** position $i \le j$ if every entry in positions $i, \dots, j-1$ is **strictly taller** than a_j.

So j sees an interval $[h(j), j]$. Here the h-**vector** is $h = (1, 2, 1, 4, 4)$: position 3 sees back to 1, position 5 only to 4.

Fact. These n intervals are **laminar**: any two are nested or disjoint.



The stable Tamari order

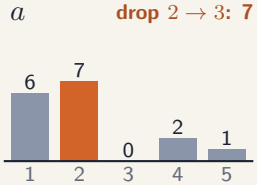
Definition

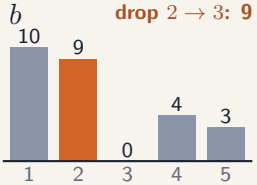
$$a \preceq b \quad \Longleftrightarrow \quad \begin{cases} a_i \le b_i & \text{for every } i, \text{ and} \\ a_i - a_j \le b_i - b_j & \text{whenever } j \text{ sees } i \text{ in } a. \end{cases}$$

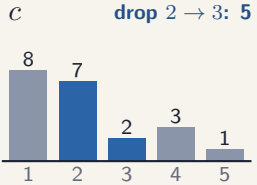
Going up, entries may grow, but every drop that a can see must stay at least as deep.

$\mathcal{ST}_n = (\mathbb{N}^n, \preceq)$ is the **stable Tamari poset**. It is *stable*: adding the same constant to every entry preserves $\preceq$. Every interval $[a, b]$ is finite, since it sits entrywise between a and b.

Example







visible pair (i, j) in a	$(1,3)$	$(2,3)$	$(4,5)$	verdict
drops in $a = (6,7,0,2,1)$	6	7	1	(reference)
drops in $b = (10,9,0,4,3)$	10	9	1	$a \preceq b$ holds
drops in $c = (8,7,2,3,1)$	6	**5**	2	$a \preceq c$ fails

c lies entrywise above a, yet the drop from position 2 to 3 shrinks from 7 to 5.

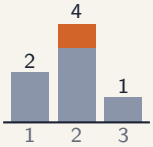
Cover relations

$$a \lessdot a + e_{[h(j),j]} \qquad (j = 1, \ldots, n)$$

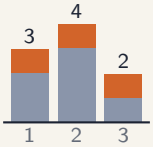
Raise by one the block that some position j sees. These n upper covers are all distinct.



$a = (2, 3, 1)$ $\qquad$ $j = 1$: $(3, 3, 1)$



$j = 2$: $(2, 4, 1)$



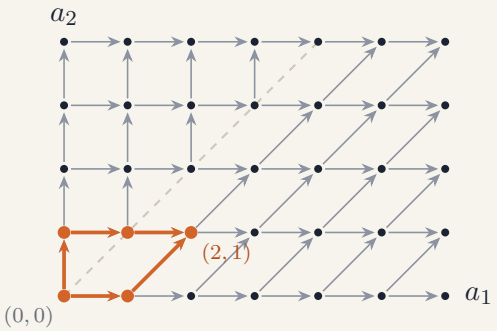
$j = 3$: $(3, 4, 2)$

The case $n = 2$

If $a_1 \leq a_2$ (on or above the diagonal), the covers step **right** or **up**.

If $a_1 > a_2$, position 2 sees position 1, so the covers step **right** or **diagonally**.

> The downset of $(2, 1)$ is our pentagon, with long side through $(0, 1)$ and $(1, 1)$.

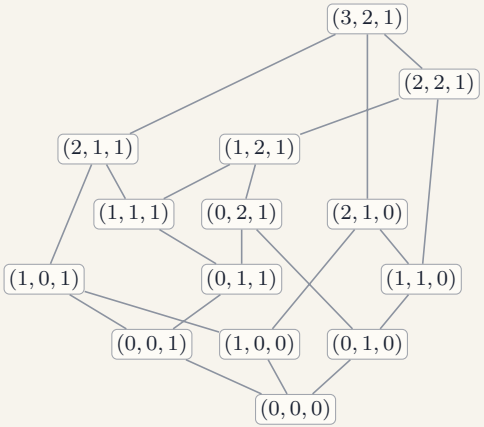


Downsets

For $b \in \mathbb{N}^n$, its **downset** is $b{\downarrow} = \{a : a \preceq b\}$.

Here $a \preceq (3,2,1)$ exactly when $(3,2,1) - a$ is weakly decreasing: these a are the area sequences of Dyck paths of semilength 4, with the final 0 dropped.

In general, $(n, n-1, \ldots, 1){\downarrow}$ is the Tamari lattice on Dyck paths of semilength $n+1$. Here, there are $14 = C_4$ elements and 68 intervals (as shown by Chapoton).



Every r-Tamari lattice lives inside

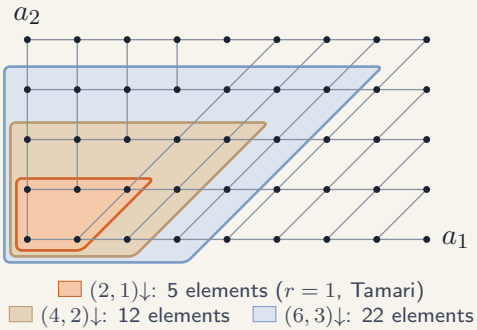
$$\tau_{r,n} = (rn,\ r(n-1),\ \ldots,\ 2r,\ r)$$

Its downset is the r-Tamari lattice on r-Dyck paths of height $n+1$:

$$\tau_{1,n}{\downarrow} \subseteq \tau_{2,n}{\downarrow} \subseteq \tau_{3,n}{\downarrow} \subseteq \cdots$$

$$\bigcup_{r\geq 1} \tau_{r,n}{\downarrow} = \mathcal{ST}_n.$$

For a partition λ (weakly decreasing), $\lambda{\downarrow} = \{a : \lambda - a \text{ is a partition}\}$ is the ν-Tamari lattice whose boundary ν is the outline of λ.



Structure inherited from the finite pieces

It is a lattice

A nested union of finite lattices, each a downset of the next. Meets and joins live in any r-Tamari piece containing both elements.

Joins are not coordinatewise

$(1,0) \vee (0,1) = (2,1)$, not $(1,1)$;
and $(1,0) \wedge (1,1) = (0,0)$, not $(1,0)$.

Intervals are classical

Every interval $[a, b]$ is isomorphic to an interval of a classical Tamari lattice.

Semidistributive and trim

Semidistributive; every interval is trim (an ungraded analogue of distributive). Not semimodular for $n \geq 2$: the pentagon sits inside.

THE PLAN

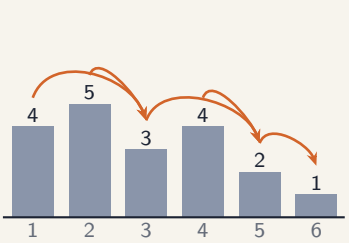
✓ Step 1: one home
a single lattice containing every r-Tamari lattice

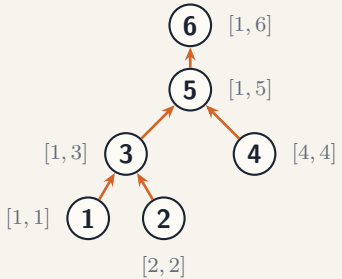
2 **Step 2: measure**
coordinates in which longest chains become sums

3 Step 3: count
all the polynomials $i_\beta^{(r)}$ in one finite formula

The forest of a sequence

Each position j points to its **parent**: the first position to its right holding a smaller entry. Positions with nothing smaller to their right are **roots**, so we get a rooted forest.





Key fact. The descendants of j are exactly the positions in $[h(j), j]$, the interval that j sees.

Comparing two sequences

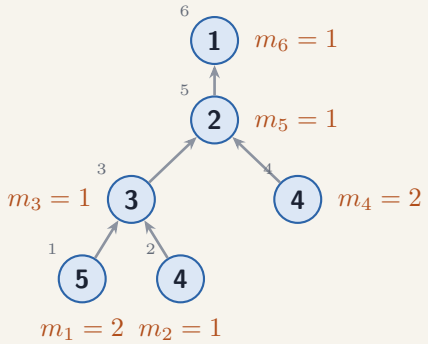
Theorem

$$b - a = \sum_{j=1}^{n} m_j \, e_{[h_a(j),j]}$$

$$a \preceq b \iff m = (m_1, \ldots, m_n) \in \mathbb{N}^n$$

On the forest of a, label each vertex by how much b exceeds a there. Then m at a vertex is its label minus its parent's label (roots keep theirs).

So $a \preceq b$ exactly when labels weakly decrease toward the roots.



$a = (4, 5, 3, 4, 2, 1)$, $b = (9, 9, 6, 8, 4, 2)$; labels $b_j - a_j$.

Height without grading

$$\mathrm{ht}(a, b) = m_1 + m_2 + \cdots + m_n$$

For $a \preceq b$, the longest chain from a to b has length equal to the coordinate sum of m. So the statistic $d(\alpha, \beta)$ in the conjecture is a sum of coordinates.

Forest example: $2+1+1+2+1+1=8$.

Pentagon: $m(0,(2,1)) = (2,1)$, so height 3: the long side.

Looking up is easy; looking down is the question

$$\sum_{b \succeq a} t^{\,\mathrm{ht}(a,b)} = \frac{1}{(1-t)^n} \qquad \text{for } \textit{every } a \in \mathbb{N}^n.$$

$b \mapsto m(a,b)$ is a bijection from the elements above a onto $\mathbb{N}^n$; height is the coordinate sum.

(The coefficient of t^k counts the b at height k above a.)

But the up-sets differ

Above $(0,0)$, nothing covers both $(1,0)$ and $(0,1)$.

Above $(0,1)$, the element $(1,2)$ covers both $(1,1)$ and $(0,2)$.

Looking down: $f_b(t) = \sum_{a \preceq b} t^{\,\mathrm{ht}(a,b)}$. When b encodes an r-Dyck path β, this is exactly $i_\beta^{(r)}(t)$, the polynomial we are after.

THE PLAN

✓ Step 1: one home
a single lattice containing every r-Tamari lattice

✓ Step 2: measure
coordinates in which longest chains become sums

3 **Step 3: count**
all the polynomials $i_\beta^{(r)}$ in one finite formula

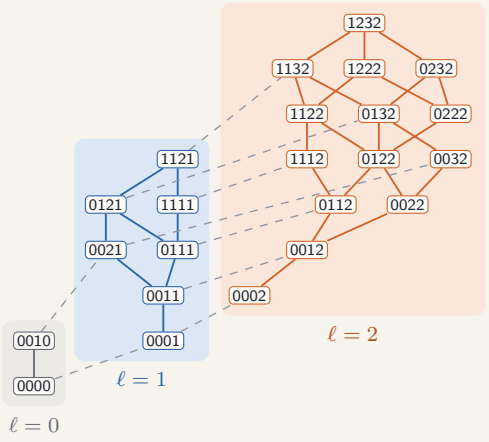
Slicing a downset

Sort the elements of $(b,k)\!\downarrow$ by their **last entry** $\ell = 0, 1, \ldots, k$.

Top layer. The elements with $\ell = k$ form the interval $[(0,k),(b,k)] \cong b\!\downarrow$.

The rest. The elements with $\ell < k$ form another downset, $(b^-, k-1)\!\downarrow$.

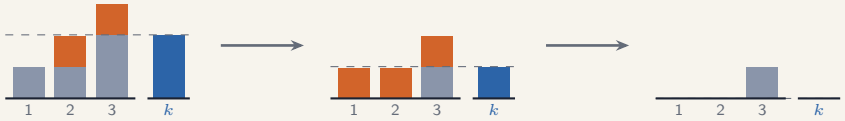
So $(b,k)\!\downarrow$ splits into $k+1$ intervals, each a copy of a smaller downset.



$(1,2,3,2)\!\downarrow$, with $abcd = (a,b,c,d)$. Layers: copies of $(0,0,1)\!\downarrow$, $(1,1,2)\!\downarrow$, $(1,2,3)\!\downarrow$. Dashed: covers between layers.

The suffix rule

$(b, k) \rightsquigarrow (b^-, k-1)$: subtract 1 from the longest suffix of b whose entries are all $\geq k$.



$(1, 2, 3 \,|\, 2)$
$|(1, 2, 3)\!\downarrow| = 14$

$(1, 1, 2 \,|\, 1)$
$|(1, 1, 2)\!\downarrow| = 7$

$(0, 0, 1 \,|\, 0)$
$|(0, 0, 1)\!\downarrow| = 2$

Counting intervals from the bottom

To track heights, sum over all intervals $[a, b]$, choosing the **bottom** a first. With $x^b = x_1^{b_1} \cdots x_n^{b_n}$ and $x_I = \prod_{i \in I} x_i$, Step 2 gives all tops at once:

$$\sum_{b \succeq a} t^{\,\mathrm{ht}(a,b)} x^b \;=\; x^a \prod_{j=1}^{n} \frac{1}{1 - t\, x_{[h_a(j),j]}}.$$

e.g. $a = (3, 1, 2, 4, 3)$, $h_a = (1, 1, 3, 4, 4)$: the product is $1/(1-tx_1)(1-tx_1x_2)(1-tx_3)(1-tx_4)(1-tx_4x_5)$.

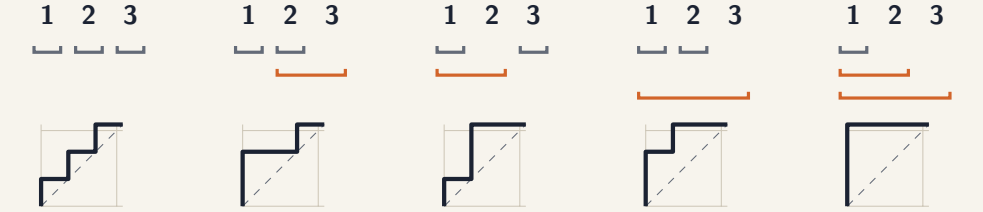
The product depends on a only through h_a. So group the bottoms by h-vector:

$$\sum_{b} f_b(t)\, x^b = \sum_{h} \Big(\sum_{a:\, h_a = h} x^a \Big) \prod_{j=1}^{n} \frac{1}{1 - t\, x_{[h(j),j]}}.$$

Which h occur, and what is the inner sum?

Which h-vectors occur?

h is the h-vector of some $a \in \mathbb{N}^n$ exactly when $h(j) \leq j$ and the $[h(j), j]$ are laminar.
For $n = 3$:



$h = (1,2,3)$	$h = (1,2,2)$	$h = (1,1,3)$	$h = (1,2,1)$	$h = (1,1,1)$
$a(h) = (0,0,0)$	$a(h) = (0,1,0)$	$a(h) = (1,0,0)$	$a(h) = (1,1,0)$	$a(h) = (2,1,0)$

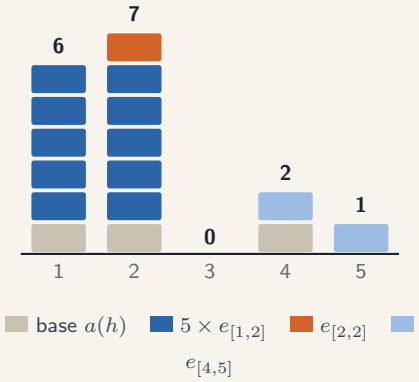
There are C_n h-vectors, one per Dyck path. The fiber $\{a : h_a = h\}$ has a $\preceq$-smallest element $a(h)$: the area sequence of that Dyck path.

Each fiber is a copy of $\mathbb{N}^n$

Every a with $h_a = h$ is, uniquely,
$a(h) + c_1 e_{I_1} + \cdots + c_n e_{I_n}$ with $c \in \mathbb{N}^n$,
where I_r is the longest interval starting at $h(r)$ on which $a(h)$ stays $\geq a(h)_r$. These vectors form a unimodular basis of $\mathbb{Z}^n$, so

$$\sum_{a:\, h_a = h} x^a = \frac{x^{a(h)}}{\prod_r (1 - x_{I_r})}.$$

Example: $h = (1, 2, 1, 4, 4)$, the h-vector of $(6, 7, 0, 2, 1)$. Then $a(h) = (1, 1, 0, 1, 0)$ and $I_1, \ldots, I_5 = [1, 2], [2, 2], [1, 5], [4, 4], [4, 5]$:
$(6, 7, 0, 2, 1) = a(h) + 5\, e_{[1,2]} + e_{[2,2]} + e_{[4,5]}$.



Payoff: every $i_\beta^{(r)}$ in one formula

$$\sum_{b\in\mathbb{N}^n} f_b(t)\,x^b \;=\; \sum_{h} \frac{x^{a(h)}}{\displaystyle\prod_{I\in\mathcal{I}(h)}(1-x_I)} \cdot \frac{1}{\displaystyle\prod_{r=1}^{n}\bigl(1-t\,x_{[h(r),r]}\bigr)}$$

Outer sum: the C_n h-vectors h.

First factor: every bottom a with $h_a = h$.

Second factor: every top $b \succeq a$; t counts height.

$$n=2: \quad \frac{1}{(1-x_1x_2)(1-x_2)(1-tx_1)(1-tx_2)} + \frac{x_1}{(1-x_1x_2)(1-x_1)(1-tx_1)(1-tx_1x_2)}$$

Check: the coefficient of $x_1^2x_2$ is $1+2t+t^2+t^3$, the pentagon again. When b encodes an r-Dyck path β, $f_b(t) = i_\beta^{(r)}(t)$ exactly; at $t=1$ it counts the downset.

THE PLAN

✓ Step 1: one home
a single lattice containing every r-Tamari lattice

✓ Step 2: measure
coordinates in which longest chains become sums

✓ Step 3: count
all the polynomials $i_\beta^{(r)}$ in one finite formula

\+ **Extras**
shellability, the Möbius function, symmetries

EL-shellability

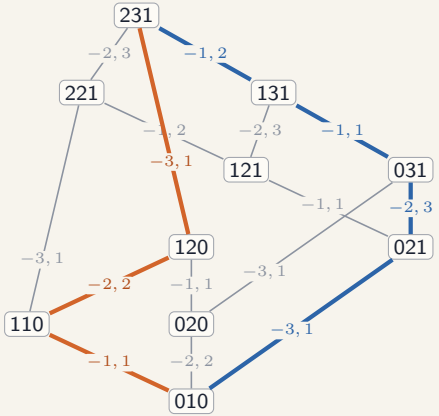
Label every cover relation. The labeling is an **EL-labeling** if every interval has exactly one maximal chain with increasing labels, and that chain is lexicographically first.

Theorem

Label a cover $a \lessdot b$ by

$$\lambda(a, b) = (-j,\ b_j),$$

where $j = \max\{i : a_i \neq b_i\}$, ordered lexicographically. This is an EL-labeling of every interval of $\mathcal{ST}_n$.



What the labeling gives

Why it works

The rising chain always raises the *last* position where it still differs from the top: the greedy chain from Step 2, so it is also a longest chain.

A chain that first raises an earlier position must raise that last position later, so its labels cannot increase.

Topology (Björner–Wachs)

Each open interval (a, b) is homotopy equivalent to a wedge of spheres, one per falling maximal chain.

Here there is **at most one** falling chain: each open interval is contractible or a single sphere, and $\mu(a, b) \in \{0, \pm 1\}$.

Tamari lattices are EL-shellable (Björner–Wachs); on r-Tamari lattices this labeling matches Mühle's up to relabeling. Trimness gives *some* EL-labeling (Liu); here it is explicit and uniform.

The Möbius function from the bottom

$$\mu(x,x) = 1, \qquad \mu(x,z) = -\sum_{x \le y < z} \mu(x,y)$$

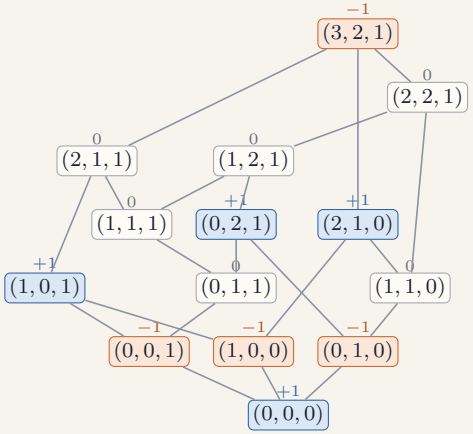
the poset version of inclusion–exclusion.

Theorem

$\mu(0,b) = 0$ unless b is a string of staircases $(k, k-1, \ldots, 1)$ separated by zeros, like $(0,2,1)$ or $(3,2,1,0,1)$. Then

$$\mu(0,b) = (-1)^{\#\{i\colon b_i > 0\}}.$$

Exactly 2^n such b: the joins of subsets of the atoms $e_1, \ldots, e_n$ (via Rota's crosscut theorem).



The Möbius function of any interval

Expand $b - a$ in the intervals seen from the **top** element b: $\quad b - a = \sum_j x_j \, e_{[h_b(j),j]}$.

$$\mu(a,b) = \begin{cases} (-1)^{|\operatorname{supp} x|} & \text{if } x \in \{0,1\}^n \text{ and } \operatorname{supp} x \subseteq D(b), \\ 0 & \text{otherwise,} \end{cases}$$

where $D(b)$ is the set of positions k at which b has a lower cover.

Proof: count falling chains

$\mu(a,b)$ counts falling chains with sign. Their changed positions increase going up, which forces $x \in \{0,1\}^n$, $\operatorname{supp} x \subseteq D(b)$.

Example

$a = (0,1,0)$, $b = (2,3,1)$: $x = (1,1,1)$ and $D(b) = \{1,2,3\}$, so $\mu = -1$. Falling chain:
$(0,1,0) \lessdot (1,1,0) \lessdot (1,2,0) \lessdot (2,3,1)$.

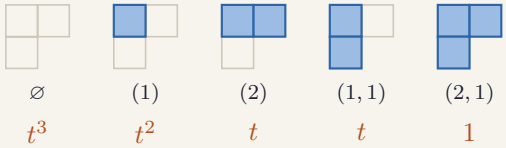
A proven symmetry: conjugate partitions

For a partition λ, the conjugate λ' transposes its Young diagram. Using Préville-Ratelle–Viennot duality:

$$\lambda{\downarrow} \;\cong\; (\lambda'{\downarrow})^{\mathrm{op}}.$$

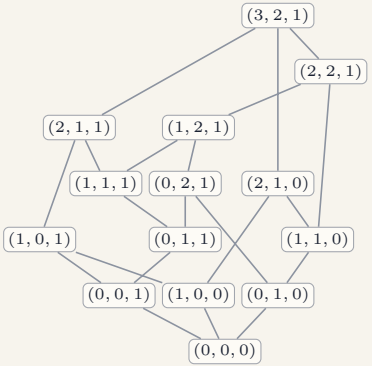
$$f_\lambda(t) = \sum_{\mu \subseteq \lambda} t^{\,|\lambda| - |\mu|}$$

Heights below λ simply count the partitions inside λ, by size.

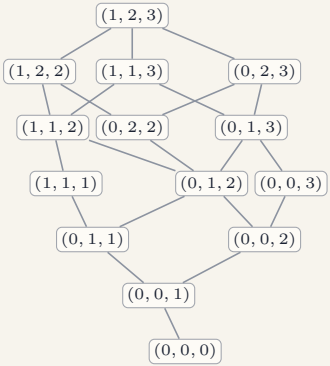


$\lambda = (2,1)$: $f_\lambda(t) = 1 + 2t + t^2 + t^3$, the pentagon seen from its top.

A conjectured symmetry: reversal



$(3,2,1)\!\downarrow$



$(1,2,3)\!\downarrow$

Different posets, same height polynomial:

$$1 + 3t + 3t^2 + 3t^3 + 2t^4 + t^5 + t^6$$

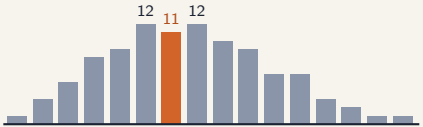
Theorem

$|b\!\downarrow| = |\operatorname{rev}(b)\!\downarrow|$ for every b, via $a \mapsto \operatorname{rev}(b - a)$.

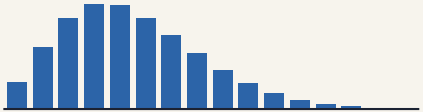
That bijection does not preserve height.

Unimodality

Two height polynomials: $f_b(t) = \sum_{a \preceq b} t^{\,\mathrm{ht}(a,b)}$ (intervals with top b) and $F_b(t) = \sum_{p \preceq q \preceq b} t^{\,\mathrm{ht}(p,q)}$ (all intervals inside $b{\downarrow}$).



$f_{(4,3,4,4)}(t)$: dips at t^6



$F_{(4,3,4,4)}(t)$: unimodal

f_b **can dip.** Stanton showed $(8,8,4,4)$ gives $\ldots, 31, 30, 31, \ldots$.

F_b **has not**, though it is *not* log-concave: $F_{(2,1)} = 5 + 5t + 2t^2 + t^3$.

Conjecture. $F_b(t)$ is unimodal for every $b \in \mathbb{N}^n$.

Why expect unimodality?

For a staircase $\tau = \tau_{r,n}$, the downset is the r-Tamari lattice and $F_\tau(t) = \sum_\beta i_\beta^{(r)}(t)$. Setting $q_2 = 1$ in the Bergeron–Préville-Ratelle conjecture predicts

$$F_\tau(t) = \text{Hilbert series of the sign-isotypic part of } \mathcal{H}_{n+1}^{(r)}, \text{ graded by } x\text{-degree.}$$

τ	$F_\tau(t)$	their computed character at $(t,1,1)$
$(2,1)$	$5 + 5t + 2t^2 + t^3$	$(s_3 + s_{11})(t,1,1)$ ✓
$(4,2)$	$12 + 16t + 14t^2 + 8t^3 + 5t^4 + 2t^5 + t^6$	$(s_6 + s_{41} + s_{22})(t,1,1)$ ✓
$(3,2,1)$	$14 + 21t + 15t^2 + 10t^3 + 5t^4 + 2t^5 + t^6$	$(s_6 + s_{41} + s_{31} + s_{111})(t,1,1)$ ✓

So F_b is (conjecturally) a graded dimension of a representation, where unimodality is often explained by an $\mathfrak{sl}_2$-action. Is there one here? A single f_b is only one summand, so no such reason applies to it.

Open questions

Reversal

Is $f_b(t) = f_{\mathrm{rev}(b)}(t)$ for every $b \in \mathbb{N}^n$? Known at $t = 1$.

Unimodality

Is $F_b(t) = \sum_{p \preceq q \preceq b} t^{\mathrm{ht}(p,q)}$ always unimodal? The top-anchored f_b can dip.

Sizes of downsets

No closed formula is known for $|b{\downarrow}|$; there is a recursion by the last entry, and the finite rational generating function.

Back to algebra

Does the finite rational formula help with the Bergeron–Préville-Ratelle conjecture, or suggest the missing statistic for the third grading q_3?

Takeaways

? **The question**
Longest chains in r-Tamari lattices feed a conjecture about trivariate diagonal harmonics.

1 **One home**
The stable Tamari lattice on $\mathbb{N}^n$ is the union of all the r-Tamari lattices.

2 **Measure**
Coordinates from the bottom turn the longest chain into a sum; from the top, the Möbius function.

3 **Count**
Grouping intervals by h-vectors gives one finite formula for every $i_\beta^{(r)}$.

Thank you! arXiv:2610.39257