



# Castelnuovo-Mumford Regularity of Toric Surfaces

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# Castelnuovo-Mumford regularity

## Definition

Let  $I \subseteq \mathbb{K}[x_0, \dots, x_n]$  be a homogeneous ideal, and consider the graded minimal free resolution

$$0 \leftarrow R/I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n+1} \leftarrow 0$$

of  $R/I$ , where  $F_i \cong \bigoplus_j R(-i-j)^{\beta_{i,i+j}}$ . The Castelnuovo-Mumford regularity (or simply regularity) is

$$\text{reg}(R/I) = \max_{i,j} \{j : \beta_{i,i+j} \neq 0\}.$$

The regularity is the index of the bottom row of the Betti table.

# Castelnuovo-Mumford regularity

## Definition

A coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^n$  is  $m$ -regular if

$$H^i(\mathcal{F}(m-i)) = 0$$

for all  $i > 0$ . The Castelnuovo-Mumford regularity (or simply regularity) is

$$\inf \{d : H^i(\mathcal{F}(d-i)) = 0 \text{ for all } i > 0\}.$$

# Monomial curves

## Definition

The monomial curve with exponents  $a_1 \leq \dots \leq a_{n-1}$  in  $\mathbb{P}^n$  is the curve  $C \subset \mathbb{P}^n$  of degree  $d = a_n$  parameterized by

$$\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^n \quad \text{with} \quad (s, t) \mapsto (s^d, s^{d-a_1}t^{a_1}, \dots, s^{d-a_{n-1}}t^{a_{n-1}}, t^d).$$

## Theorem (L'vovsky; 1996)

Let  $A = (0, a_1, \dots, a_n)$  be a sequence of non-negative integers such that the g.c.d. of the  $a_j$ 's equals 1, and let  $C$  be the corresponding monomial curve. Then  $C$  is  $m$ -regular, where

$$m = \max_{1 \leq i < j \leq n} \{(a_i - a_{i-1}) + (a_j - a_{j-1})\},$$

i.e.,  $m$  is the sum of the two largest gaps in the semigroup generated by  $A$ .

## Example (sporadic)



$$A = \begin{pmatrix} 0 & 3 & 5 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 & 5 & 7 \\ 7 & 4 & 2 & 0 \end{pmatrix}$$

$$\varphi(s, t) = (s^7, \underset{x_0}{s^3 t^4}, \underset{x_1}{s^5 t^2}, \underset{x_2}{t^7})$$

$$I_C = \langle x_2^2 - x_1 x_3, \quad x_1^3 x_2 - x_0^2 x_3^2, \quad x_1^4 - x_0^2 x_2 x_3 \rangle$$

$$0 \leftarrow R/I_C \leftarrow R \leftarrow \frac{R(-2)}{\oplus_{R(-4)^2}} \leftarrow R(-5)^2 \leftarrow 0$$

$$\text{reg}(I_C) = 4 \leq (a_1 - a_0) + (a_2 - a_1) = 3 + 2 = 5$$

```
i1: kk = ZZ/32749;
```

```
i2: I = monomialCurveIdeal(kk[x_0..x_3], {3,5,7})
```

```
o2 = ideal (x2 - x3x1, x3x2 - x2x3, x4 - x2x3x1)  
          2      3      2 2    4      2  
          1 3      1 2      0 3      1      0 2 3
```

```
i3 : print betti res I
```

```
0 1 2
```

```
total: 1 3 2
```

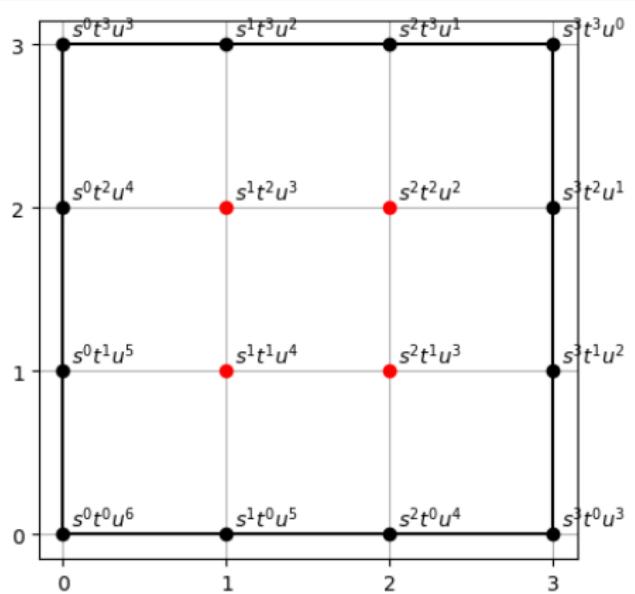
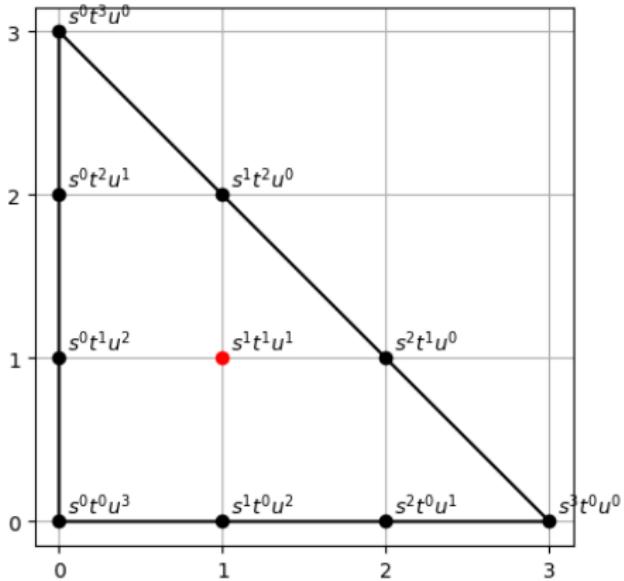
```
0: 1 . .
```

```
1: . 1 .
```

```
2: . . .
```

```
3: . 2 2
```

# Toric surfaces



# Eisenbud-Goto

## Definition

A polytope  $P$  is  $k$ -normal if the map

$$\underbrace{P + P + \dots + P}_{k \text{ times}} \longrightarrow kP$$

is surjective. Define  $k_P$  to be the smallest  $k$  such that  $P$  is  $k$ -normal.

## Conjecture (Eisenbud-Goto, 1984)

For a smooth projective variety  $X$ ,

$$\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}(X) + 1.$$

In particular, for a projective toric variety coming from a polytope  $P$ ,

$$k_P \leq \mathrm{Vol}(P) - |P| + \dim P - 1.$$

# What has been done?

Theorem (Lazarsfeld, 1997)

*Every smooth, projective surface satisfies the Eisenbud-Goto conjecture.*

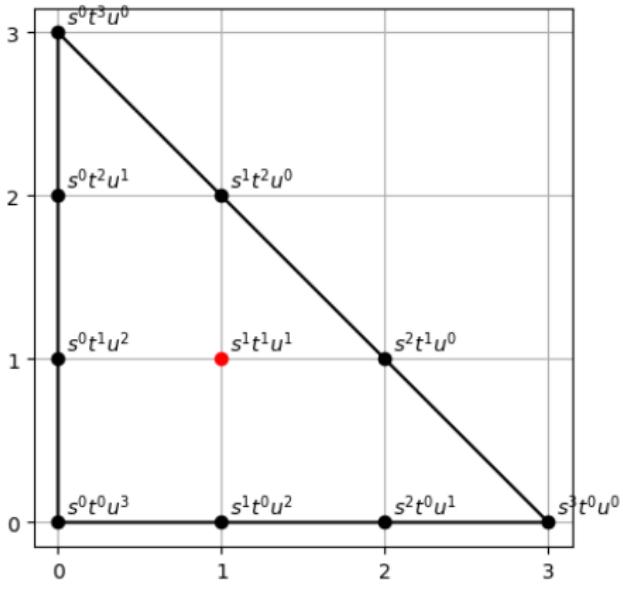
Theorem (Koelman, 1993)

*For a lattice polygon  $P$ , the ideal  $I_P$  is generated by quadric and cubic binomials. Moreover, all of the minimal generators of  $I_P$  are quadratics if and only if  $|\partial P| > 3$ .*

Theorem (Schenck, 2004; Hering, 2006)

*If  $P$  has nonempty interior, then the index where  $\beta_{i,i+2}$  is first nonzero is  $|\partial P|$ .*

# Setup



$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

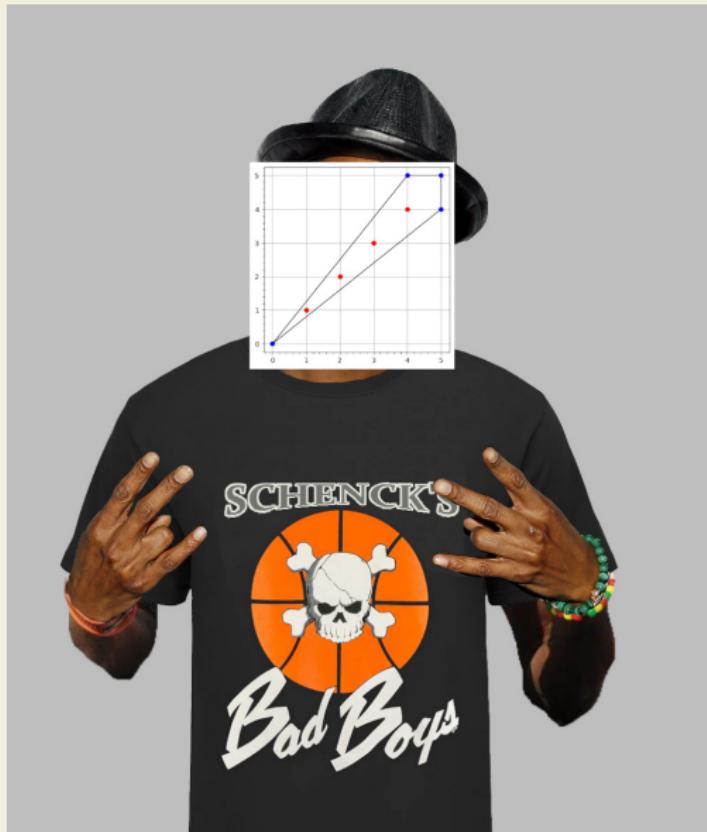
$\downarrow \text{Conv}(A) \setminus \text{Int}(A)$

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

$\downarrow \text{homogenize}$

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 2 & 1 \\ 3 & 2 & 1 & \cdots & 0 & 1 & 2 \end{pmatrix}$$

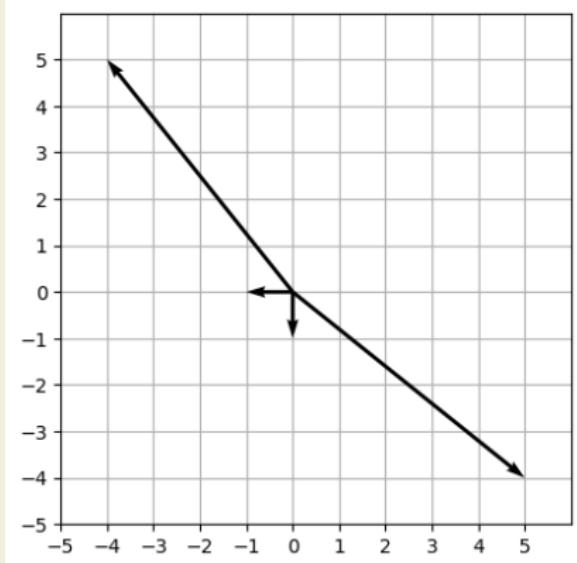
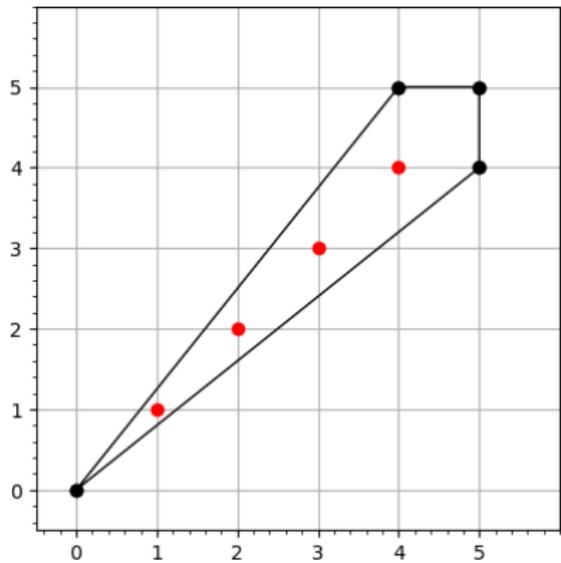
# “Bad Boy”



# “Bad Boy”

*In general, the regularity can be arbitrarily large by using*

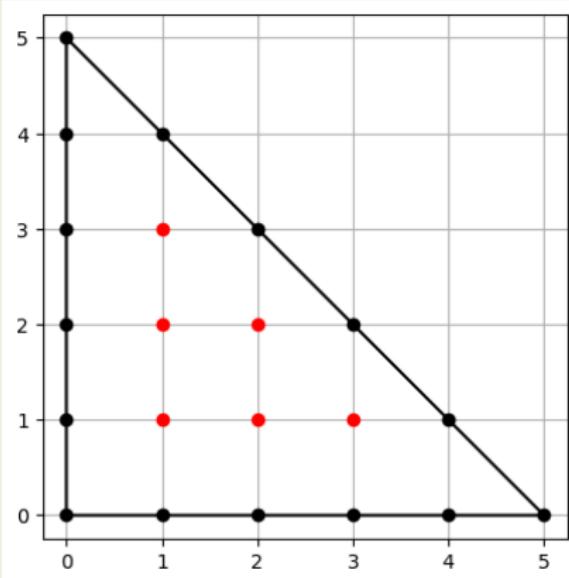
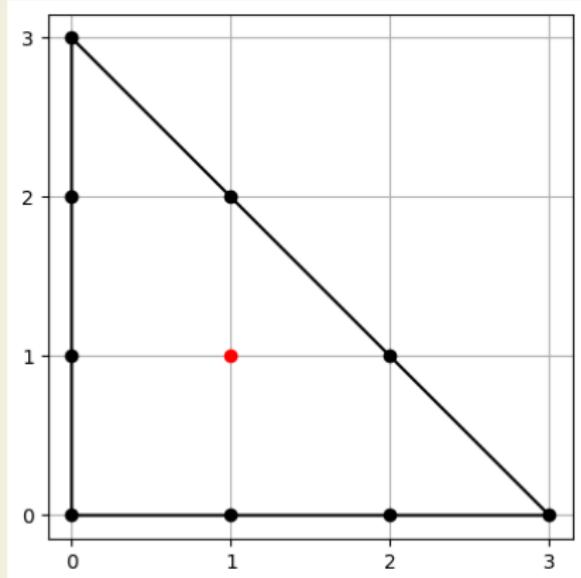
$$A = \begin{pmatrix} 0 & d & d-1 & d \\ 0 & d-1 & d & d \end{pmatrix}.$$



# Hollow triangle

## Definition

Suppose  $A = \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$ . The hollow triangle of length  $k$  is  $\Delta^k := \tilde{A}$ .



## Hollow triangle data

```
+-----+
|   |
|   |       0 1 2 3
| 2 | total: 1 6 8 3
|   |       0: 1 . .
|   |       1: . 6 8 3
|   |
+-----+
|   |
|   |       0   1   2   3   4   5   6   7   8
| 3 | total: 1 17 53 91 108 83 37 9 1
|   |       0: 1  .  .  .  .  .  .  .  .
|   |       1: . 17 43 36   8  .  .  .  .
|   |       2: .  . 10 55 100 83 37 9 1
|   |
+-----+
```

## Hollow triangle data

```
+-----+
| |          0   1   2   3   4   5   6   7   8   9   10  11
| 4 | total: 1  33  153  525 1356 2178 2205 1486 675  201  36   3
| |          0: 1   .   .   .   .   .   .   .   .   .   .   .
| |          1: .   33  123  144   30   .   .   .   .   .   .   .
| |          2: .   .   30  381 1326 2178 2205 1486 675  201  36   3
| |
+-----+
```

```
+-----+
| |          0   1   2   3   4   5   6   7   8   9
| 5 | total: 1  54  389 2028 7845 18957 30393 34672 29106 18162
| |          0: 1   .   .   .   .   .   .   .   .   .
| |          1: .   54  266  462  174   15   .   .   .   .
| |          2: .   .   123 1566 7671 18942 30393 34672 29106 18162
| |
+-----+
```

# Results

## Lemma

For all  $d \geq 2$ ,  $(R/I_{\Delta^k})_d = (\overline{R/I_{\Delta^k}})_d$ .

## Theorem

For all  $k \geq 2$ ,  $\text{reg}(\Delta^k) = 2$ .

# Results

## Lemma

For all  $d \geq 2$ ,  $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$ .

## Theorem

For all  $k \geq 2$ ,  $\text{reg}(\square^k) = 2$ .

# Proof sketch of theorem

- Use the short exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_{\square^k}(d) \rightarrow \mathcal{O}_{\mathbb{P}^{4k-1}}(d) \rightarrow \mathcal{O}_{\square^k}(d) \rightarrow 0$$

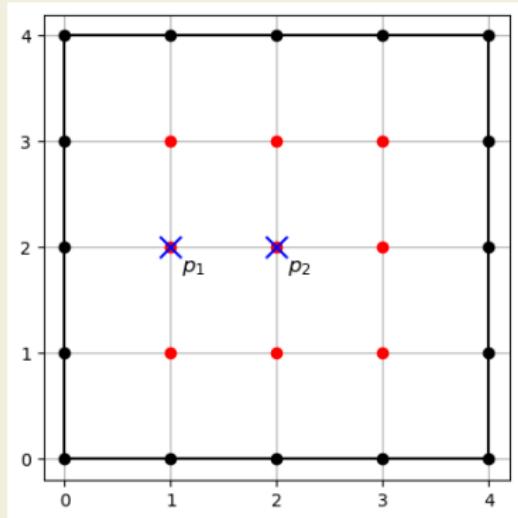
to eventually get a short exact sequence

$$0 \rightarrow R/I_{\square^k} \rightarrow \overline{R/I_{\square^k}} \rightarrow N \rightarrow 0.$$

- By the lemma,  $N$  is generated is only generated by degree 1 monomials.
- $\text{reg}(R/I_{\square^k}) \leq \max(\text{reg}(\overline{R/I_{\square^k}}), N) = \text{reg}(\overline{R/I_{\square^k}}) = 2$ .

# Proof sketch of lemma

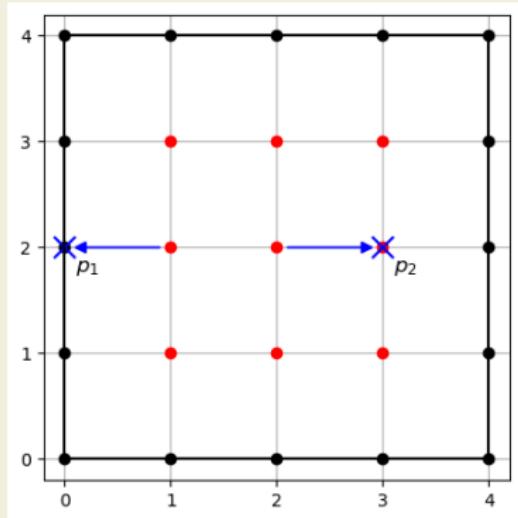
- Showing  $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$  for  $d \geq 2$  amounts to a computation with the lattice points of  $\overline{\square^k}$ .
- We are done if for any  $p_1, p_2 \in \overline{\square^k}$ , we can write  $p_1 + p_2 = q_1 + q_2$  with  $q_1, q_2 \in \square^k$ .



$$p_1 + p_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$$

# Proof sketch of lemma

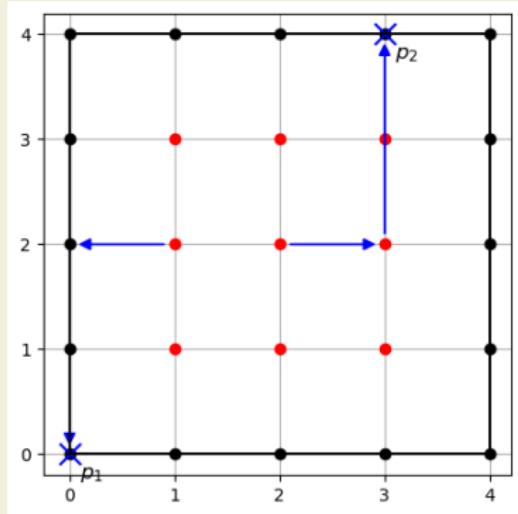
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$$\begin{aligned} p_1 + p_2 &= \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \end{aligned}$$

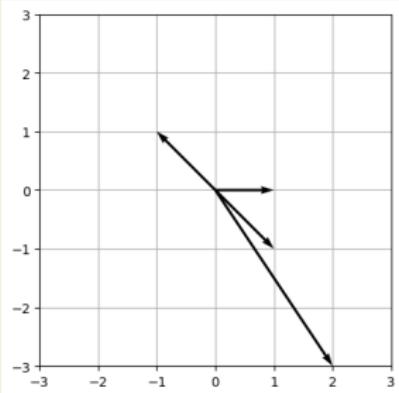
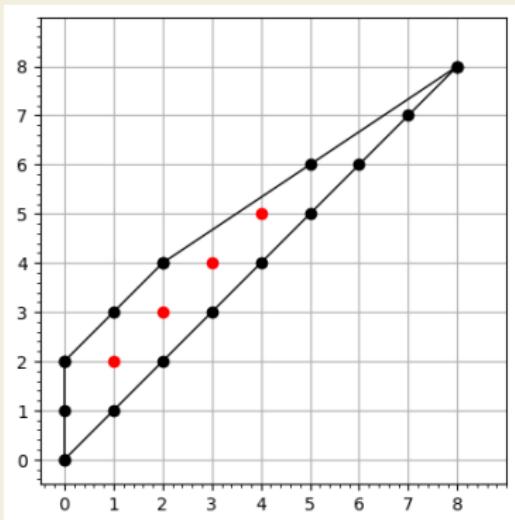
# Proof sketch of lemma

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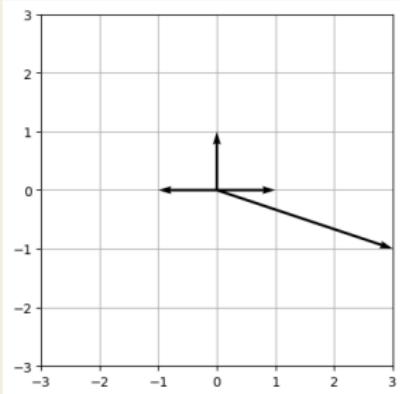
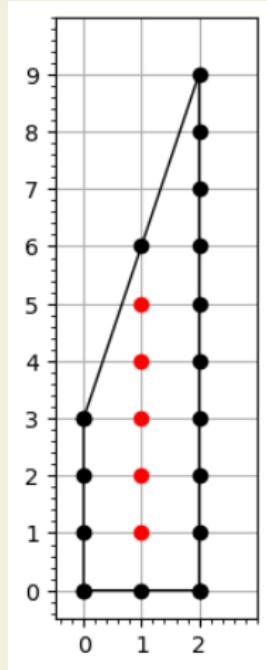
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# Smooth is not enough



	0	1	2	3	4	5	...
total:	1	54	385	1462	3608	6456	...
0:	1	.	.	.	.	.	...
1:	.	52	280	730	1128	1050	...
2:	.	.	81	600	2040	4416	...
3:	.	2	24	132	440	990	...

# Smooth is not enough



0	1	2	3	4	5	...	
total:	1	74	633	2883	8593	18953	...
0:	1	.	.	.	.	.	...
1:	.	73	486	1627	3388	4620	...
2:	.	1	144	1218	4983	13541	...
3:	.	.	3	38	222	792	...