Incidence toric ideals and three-point functions

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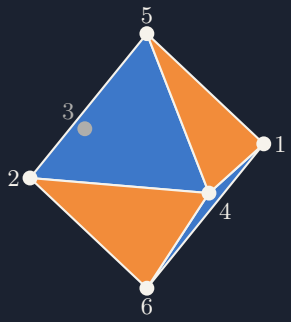
SIAM TX-LA Sectional Meeting

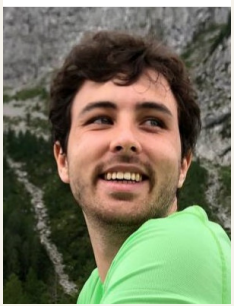


The Crew

Barbara Betti

Sean Grate

Thiago Holleben



Flavio Salizzoni

Three-point functions

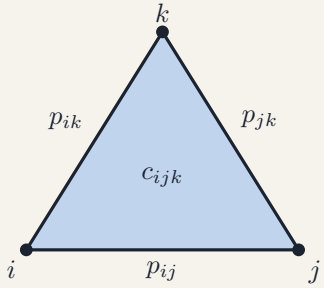
Massless momenta $p_1, \ldots, p_n \in \mathbb{C}^d$, with Mandelstam invariants $p_{ij} = p_i \cdot p_j$ (Lorentzian, $p_{ii} = 0$).

Three-point functions, all scaling dimensions 2:

$$C_{ijk} = \frac{1}{p_{ij}\, p_{jk}\, p_{ik}}.$$

Question

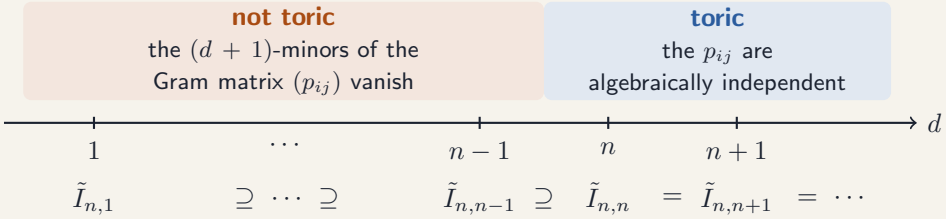
Which algebraic relations hold among the C_{ijk}?



$$c_{ijk} := C_{ijk}^{-1} = p_{ij}\, p_{jk}\, p_{ik}$$

The answer depends on the dimension

Let $\tilde{I}_{n,d} \subseteq \mathbb{C}[c_{ijk}]$ be the ideal of relations when the momenta live in $\mathbb{C}^d$.



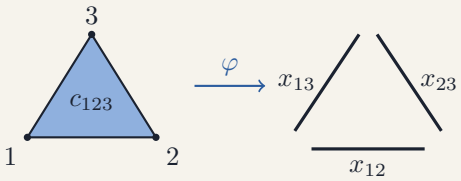
Toric for $d \geq n$: Rajan–Sverrisdóttir–Sturmfels (2024). Gram-matrix minors for $d < n$: El Maazouz–Sturmfels–Sverrisdóttir (2024).

Incidence toric ideals

Definition

$$\varphi\colon\ \mathbb{K}\Big[c_\kappa : \kappa \in \tbinom{[n]}{k}\Big] \longrightarrow \mathbb{K}\Big[x_\tau : \tau \in \tbinom{[n]}{t}\Big], \qquad c_\kappa \longmapsto \prod_{\tau\subset\kappa,\ |\tau|=t} x_\tau$$

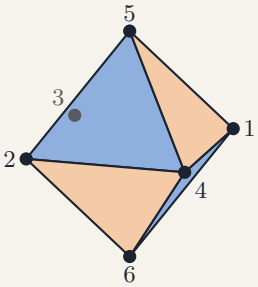
$I_{n,k,t} := \ker\varphi$, with exponent matrix $A_{n,k,t}$ and polytope $P_{n,k,t} := \operatorname{conv}(\text{columns of } A_{n,k,t})$.



$k = 3,\ t = 2$: $\tilde{I}_{n,n} = I_{n,3,2}$

- $t = k - 1$: a signless boundary map $\rightsquigarrow$ topology
- $t = 1$: the hypersimplex $\Delta(k, n)$
- homogeneous: every column has $\binom{k}{t}$ ones

Running example: $n = 6$, $k = 3$, $t = 2$



antipodal pairs $\{1,2\}$, $\{3,4\}$, $\{5,6\}$

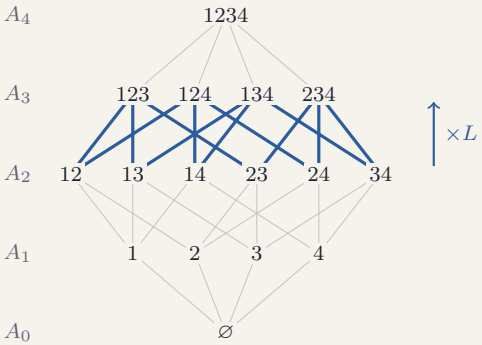
30	**5**	**162**
minimal generators	codimension	degree

15 *octahedral quartics*, one for each octahedron on $[6]$:

$$c_{135}\,c_{146}\,c_{236}\,c_{245} \;-\; c_{136}\,c_{145}\,c_{235}\,c_{246}$$

The two sides are the blue and orange facets. Plus 15 sextics.

A Lefschetz map in disguise



$n = 4$: monomials of $A \leftrightarrow$ subsets. Blue: $\times L\colon A_2 \to A_3$.

$$A = \mathbb{K}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$$

and $L = x_1 + \cdots + x_n$. Multiplying by L adds one element in every possible way, so

$$\times L^{k-t}\colon A_t \to A_k \quad \text{is} \quad (k-t)!\, A^{\mathsf{T}}_{n,k,t}$$

(cf. Holleben 2025).

Theorem (Stanley 1980)

In characteristic 0, monomial complete intersections have the strong Lefschetz property.

Proposition

$A_{n,k,t}$ has full rank:

$$\operatorname{rank} A_{n,k,t} = \min\left\{\binom{n}{t}, \binom{n}{k}\right\}.$$

Proof. $\times L^{k-t} = (k-t)!\, A_{n,k,t}^{\mathsf{T}}$ has full rank by Stanley (1980).

Consequences. $I_{n,k,t} \neq 0$ iff $\binom{n}{t} < \binom{n}{k}$, and then $\operatorname{ht} I_{n,k,t} = \binom{n}{k} - \binom{n}{t}$, $\dim V(I_{n,k,t}) = \binom{n}{t} - 1$.

Example. $I_{6,3,2}$ has height $20 - 15 = 5$.

Characteristic p and the polytope

Theorem (Jonsson Kling 2026)

In characteristic p, the map $\times L^{k-t}\colon A_t \to A_k$ has full rank if and only if $p > \min(k,\, n-t)$.

Proposition

Every prime $p \leq \min(k,\, n-k+1)$ divides the normalized Euclidean volume of $P_{n,k,k-1}$.

Corollary

The product of all primes $p \leq k$ divides $\mathrm{Vol}(P_{2k-1,k,k-1})$, so the volume is not a polynomial in n, k, t.

Caveat. Euclidean lattice, not the lattice of $A_{n,k,t}$: this is not $\deg V(I_{n,k,t})$, which remains open.

Binomials are null designs

Definition

$f\colon \binom{[n]}{k} \to \mathbb{Z}$ is a *null t-design* if $\sum_{F \supseteq X} f(F) = 0$ for every t-subset X.

$$\ker_{\mathbb{Z}} A_{n,k,t} \;\cong\; \{\text{null } t\text{-designs}\}, \qquad f \longmapsto c^{f_+} - c^{f_-}$$

Theorem (Deza–Frankl–Singhi 1983)

A nonzero null t-design takes at least 2^t positive values.

So every binomial in $I_{n,k,t}$ has degree at least 2^t.

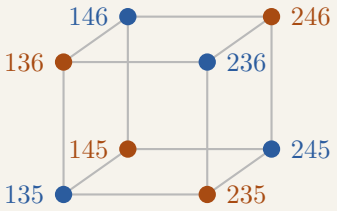
Pods: the degree bound is sharp

Theorem (Graver–Jurkat 1973; Graham–Li–Li 1980)

Null t-designs are spanned by the *pods*

$$\prod_{m=0}^{t}(x_{i_{2m+1}}-x_{i_{2m+2}})\prod_{m=2t+3}^{k+t+1}x_{i_m}.$$

A pod has exactly 2^t positive terms, so the lowest degree in $I_{n,k,t}$ is exactly 2^t.



$(x_1-x_2)(x_3-x_4)(x_5-x_6)$: terms are cube vertices, sign is parity — the octahedral quartic.

Theorem

$P_{n,k,t}$ is $(2^t - 1)$-neighborly,
and not 2^t-neighborly.

Proof. A non-face F yields $v \in \ker A_{n,k,t}$ with $\operatorname{supp}(v_+) \subseteq F$, so $|F| \geq 2^t$ by Deza–Frankl–Singhi (1983). The positive half of a pod is a non-face of size exactly 2^t.

Example. $P_{6,3,2}$ has 20 vertices in dimension 14: every 3 vertices span a face, but the 4 blue triangles of an octahedron do not.

Octahedra generate

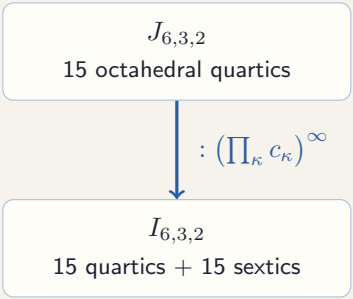
Pods are the two-row Specht polynomials (Morita–Wachi–Watanabe 2009), i.e. orientations of hyperoctahedra.

Theorem (Spink 2020)

$\ker_{\mathbb{Z}} A_{n,k,t}$ is generated by orientations of hyperoctahedra.

So the *octahedral ideal* $J_{n,k,t}$ they generate satisfies

$$I_{n,k,t} = J_{n,k,t} : \Big(\prod_{\kappa} c_\kappa\Big)^{\infty}.$$



Do the extra components matter?

Question

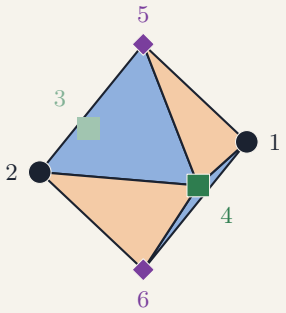
Do the components of $V(J_{n,k,t})$ in the coordinate hyperplanes have lower dimension, so that $\deg V(I_{n,k,t}) = \deg V(J_{n,k,t})$?

The smallest case $(6,3,2)$ in Macaulay2:

	codim	degree
$V(J_{6,3,2})$	5	162
$V(I_{6,3,2})$	5	162

Yes here: no extra component of the top dimension.

Balanced complexes



Colour classes $\{1,2\}$, $\{3,4\}$, $\{5,6\}$; facets 2-coloured blue/orange.

For $t = k-1$, $A_{n,k,k-1}$ is the top boundary map of the simplex with the signs forgotten.

Definition (Stanley 1979)

Δ is *balanced* if its vertices can be k-coloured so that every facet sees each colour once.

Lemma (Klee–Novik 2016)

A balanced normal pseudomanifold without boundary, of dimension ≥ 2, is orientable iff its facet–ridge graph is bipartite.

Theorem

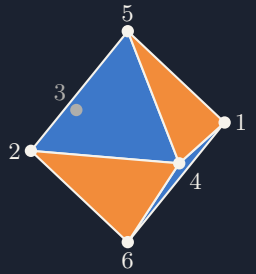
Balanced orientable pseudomanifolds give Graver basis elements.

Let $\Delta = \langle \sigma_1, \ldots, \sigma_s \rangle$ be a balanced, orientable, normal $(k-1)$-pseudomanifold without boundary on $[n]$, $k \geq 3$, with orientation ε. Then

$$\prod_{\varepsilon_i = 1} c_{\sigma_i} \; - \prod_{\varepsilon_j = -1} c_{\sigma_j}$$

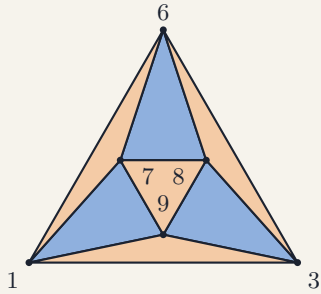
lies in the Graver basis of $I_{n,k,k-1}$.

Proof idea. Order vertices by colour: signs of ∂ depend only on the deleted colour, so $D\partial$ is a submatrix of $A_{n,k,k-1}$, with kernel $\tilde{H}_{k-1}(\Delta; \mathbb{Z}) = \mathbb{Z}\varepsilon$.



The octahedron gives the quartics.

New Graver elements from cross-flips



This disk replaces the triangle 136 of the octahedron.

Cross-flips (Izmestiev–Klee–Novik 2017) preserve balancedness and topology.

Octahedron $\rightsquigarrow$ balanced 2-sphere with 9 vertices and 14 triangles $\rightsquigarrow$

$$c_{146}\,c_{236}\,c_{135}\,c_{245}\,c_{678}\,c_{179}\,c_{389} - c_{789}\,c_{167}\,c_{368}\,c_{139}\,c_{246}\,c_{145}\,c_{235}$$

a degree-7 Graver element of $I_{9,3,2}$.

Open questions

Degree

Compute $\deg V(I_{n,k,t})$.

Octahedra vs. toric

Is $\deg V(I) = \deg V(J)$ in general? Yes for $(6, 3, 2)$.

Non-toric

Find bases for $\tilde{I}_{n,d}$ with $d < n - 1$.

Lefschetz
rank, dimension, volume
arithmetic

Null designs
degree bound, sharp
neighborliness

Topology
Graver elements from
balanced spheres