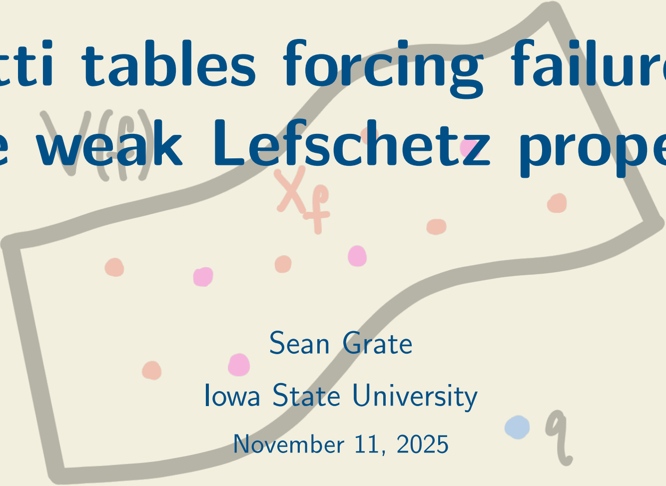
Betti tables forcing failure of the weak Lefschetz property



Sean Grate

Iowa State University

November 11, 2025

The crew

Hal Schenck
(Auburn University)

Setup

$\mathbb{K}$ is a field

$R = \mathbb{K}[x_0, \ldots, x_n]$

$I \subseteq R$ is a homogeneous ideal

A is a standard-graded Artinian $\mathbb{K}$-algebra

- Frequently, $A = R/I$
- Sometimes, $A = R/(I, L)$ where L is a linear form

Example

$A = \mathbb{K}[x, y, z]/(x^2, y^2, z^3)$

d	0	1	2	3	4	5
gens	1	x y z	xy xz yz z^2	xyz xz^2 yz^2	xyz^2	0
dim	1	3	4	3	1	0

If $I_\Delta \subseteq R$ is a Stanley-Reisner ideal, then $I_\Delta + (x_i^2 \mid 1 \leq i \leq n)$ is Artinian.

Lefschetz properties

All elements of A are zero-divisors. So, the best we can hope for is that there exists $f \in R$ such that $\times f$ is either injective or surjective (full rank) in all degrees.

Definition

$A = R/I$ has the weak Lefschetz property (WLP) if $A_d \xrightarrow{\times \ell} A_{d+1}$ is full rank for all $d \geq 0$. If $A_d \xrightarrow{\times \ell^k} A_{d+k}$ is full rank for all $d \geq 0$ and all $k \geq 0$, then A has the strong Lefschetz property (SLP).

"Who the hell cares?"

Theorem (Fröberg, 1985)

Let $f_1, \ldots, f_s$ be generic forms in $R = \mathbb{K}[x_1, \ldots, x_n]$ with degrees $d_1, \ldots, d_s$, respectively, and set $I = (f_1, \ldots, f_s)$. Then

$$HS_{R/I}(t) = \left[\frac{\prod_{i=1}^{s}(1-t^{d_i})}{(1-t)^n}\right]$$

Example

Take five quadrics in four variables:

$$\frac{(1-t^2)^5}{(1-t)^4} = 1 + 4t + 5t^2 - 5t^4 - 4t^5 + O(t^6)$$

$$\mathrm{HS}_{R/I}(t) = 1 + 4t + 5t^2$$

Geometric combinatorics

Theorem (Billera-Lee, 1981; Stanley, 1985)

A sequence of non-negative integers $(f_0, \ldots, f_d)$ is the f-vector of a simple polytope if and only if

(1) $h_i = h_{d-i}$ *for all* $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$, *(Dehn-Sommerville relations)*

(2) $g_i \geq 0$ *for all* $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$, *(upper bound conjecture)*

(3) $(g_1, \ldots, g_{\lfloor \frac{d}{2} \rfloor})$ *is a Macaulay vector.*

Geometric combinatorics

Theorem (Hard Lefschetz Theorem)

For a smooth n-dimensional projective variety X, the cup product of the k-th power of the cohomology class of a hyperplane yields and isomorphism between $H^{n-k}(X)$ and $H^{n+k}(X)$.

Corollary

Multiplication by a general linear form is injective up to degree n and surjective in higher degrees.

Geometric combinatorics

Theorem (Braden-Huh-Matherne-Proudfoot-Wang, 2020)

Let M be a matroid with $\mathcal{L}$ its lattice of flats. For all $k \leq j \leq rank(M) - k$,

(1) $|\mathcal{L}^k(M)| \leq |\mathcal{L}^j(M)|$;

(2) there is an injective map $\iota\colon \mathcal{L}^k \to \mathcal{L}^j$ *satisfying* $F \leq \iota(F)$ *for all* $F \in \mathcal{L}^k(M)$.

Algebraic Geometry	**Commutative Algebra**
Chow ring of a matroid	Artinian algebra
$\downarrow$	$\downarrow$
Hodge-Riemann relations	
$\downarrow$	$\downarrow$
Hard Lefschetz Theorem	Lefschetz properties
$\downarrow$	$\downarrow$
Log-concavity	Unimodality

WLP is hard

Theorem (Brenner-Kaid, 2010)

Let char$(\mathbb{K}) = 2$. *Then* $A = \mathbb{K}[x, y, z]/(x^d, y^d, z^d)$ *has the WLP if and only if* $d = \lfloor \frac{2^k+1}{3} \rfloor$ *for some* $k > 0$.

Theorem (Harbourne-Schenck-Seceleanu, 2011)

Let $I = (L_1^t, \ldots, L_n^t) \subset \mathbb{K}[x_1, \ldots, x_4]$ *with* $L_i \in R_1$ *generic. If* $n \in \{5, 6, 7, 8\}$, *then WLP fails, respectively, for* $t \geq \{3, 27, 140, 704\}$.

Open Problem

Does every complete intersection in four or more variables have the WLP?

Start with geometry

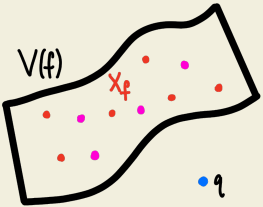
Theorem (Grate-Schenck, 2024)

Let $X_f \subset \mathbb{P}^n$ be a finite set of distinct points on a unique hypersurface $\mathbf{V}(f)$ with $\deg(f) = d$ such that $\mathbf{I}(X_f)_d = (f)$. Choose $q \notin \mathbf{V}(f)$ so that if $X := X_f \cup \{q\}$, then

(1) $\mathbf{I}(X_d) = 0$, *and*

(2) $\dim_{\mathbb{K}}(\mathbf{I}(X)_{d+1}) = n$.

Then A_X does not have the WLP. In particular, $A_d \xrightarrow{\times \ell} A_{d+1}$ does not have full rank.



Proof (sketch)

(1) (not injective)

(i)

$$\mathbf{I}(X)_{d+1} = (f) \cap (x_1, \ldots, x_n) \quad \Longleftarrow \quad \substack{\mathbf{I}(X)_{\leq d} = (f) \\ \mathbf{I}(\{q\}) = (x_1, \ldots, x_n)}$$

(ii) Get Artinian reduction $A = R/\mathbf{I}(X) + (x_0)$

(iii) $\ell \cdot f = 0$, so $A_d \xrightarrow{\times \ell} A_{d+1}$ is not injective

(2) (not surjective)

(i) Dimension count

$$\dim_{\mathbb{K}}(A_d) = \binom{n+d-1}{n-1} < \binom{n+d}{n-1} - n \quad (n \geq 3)$$

Betti tables

Given an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, a (minimal) free resolution "approximates" R/I. The Betti table records the ranks of the summands appearing in the free resolution.

Example

$R = \mathbb{K}[x, y, z]$ and $I = \langle x^2, y^2, z^3 \rangle$

$$0 \leftarrow R/I \leftarrow R \xleftarrow{\begin{bmatrix} x^2 & y^2 & z^3 \end{bmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \xleftarrow{\begin{bmatrix} y^2 & z^3 & 0 \\ -x^2 & 0 & z^3 \\ 0 & -x^2 & -y^2 \end{bmatrix}} \begin{matrix} R(-4) \\ \oplus \\ R(-5)^2 \end{matrix} \xleftarrow{\begin{bmatrix} z^3 \\ -y^2 \\ x^2 \end{bmatrix}} R(-7) \leftarrow 0$$

Example (continued)

Example

$R = \mathbb{K}[x, y, z]$ and $I = \langle x^2, y^2, z^3 \rangle$

$$0 \leftarrow R/I \leftarrow R \leftarrow \begin{array}{c} R(-(1{+}1))^2 \\ \oplus \\ R(-(1{+}2)) \end{array} \leftarrow \begin{array}{c} R(-(2{+}2)) \\ \oplus \\ R(-(2{+}3))^2 \end{array} \leftarrow R(-(3+4)) \leftarrow 0$$

$$\mathrm{Betti}(R/I) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & 2 & . & . \\ 2 & . & 1 & 1 & . \\ 3 & . & . & 2 & . \\ 4 & . & . & . & 1 \end{array}.$$

Getting back to geometry

Example

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \subset \mathbb{P}^3$$

$$\mathbf{I}(X) = (x_0x_1, x_0x_2, x_0x_3, x_1^2x_2 - x_1x_2^2, x_1^2x_3 - x_1x_3^2, x_2^2x_3 - x_2x_3^2)$$

$$\mathrm{Betti}(R/I) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & 3 & 3 & 1 \\ 2 & . & 3 & 4 & 1 \\ 3 & . & . & 1 & 1 \end{array}.$$

Koszul tails

Definition

*A Betti table B has an *(n,d)-Koszul tail* if it has an upper-left principal block of the form*

$$\begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & \dots & n-2 & n-1 & n & n+1 \\ \hline 0 & 1 & . & . & . & \dots & . & . & . & * \\ 1 & . & . & . & . & \dots & . & . & . & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & * \\ d-1 & . & . & . & . & \dots & . & . & . & * \\ d & . & n & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} & n & 1 & * \\ d+1 & * & * & * & * & * & * & * & * & * \end{array}.$$

*If B is has an (n,d)-Koszul tail and is the Betti table for an Artinian ring $\mathbb{K}[x_1,\dots,x_n]/I$, then we say B has a *maximal* (n,d)-Koszul tail.*

$(3,3)$-Koszul tail example

Example

Consider $X_f \subset \mathbb{P}^4$ lying on $\mathbf{V}(f)$ with $\deg(f) = 3$ (i.e., 34 points plus 31 extra points). Take five points X_Q lying off of $\mathbf{V}(f)$, but on a codimension 3 linear space. Set $X := X_f \cup X_Q$.

$$A_X = R/(\mathbf{I}(X), L)$$

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	.	.	.	.
3	.	3	3	1	.
4	.	46	119	102	28
5	.	.	.	.	1

(3, 3)-Koszul tail

$$A'_X = R/(\mathbf{I}(X), L, L')$$

	0	1	2	3
0	1	.	.	.
1	.	.	.	.
2	.	.	.	.
3	.	3	3	1
4	.	15	27	12

Maximal (3, 3)-Koszul tail

Getting back to Lefschetz properties

Theorem (Grate-Schenck, 2024)

An Artinian algebra $A = R/I$ whose Betti table has a maximal (n, d)-Koszul tail does not have the WLP; the map $A_d \xrightarrow{\times \ell} A_{d+1}$ is not full rank.

Corollary

If $T = \mathbb{K}[x_0, \dots, x_n]/I$ is Cohen-Macaulay of dimension m, and T has a maximal $(n - m, d)$-Koszul tail, then the Artinian reduction of T does not have the WLP.

Corollary

If $A = \mathbb{K}[x_1, \dots, x_{m+n}]/I$ is Artinian with an (n, d)-Koszul tail, and there exists a sequence of linearly independent linear forms $I_L = (l_1, \dots, l_m)$ such that A/I_L has the same top row Betti table as A, then A/I_L does not have the WLP.

Is it all about the socle?

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \subset \mathbb{P}^3$$

$$\mathbf{I}(X) = (x_0x_2, x_1x_2, x_0x_3, x_1x_3, x_0^2x_1 - x_0x_1^2, x_2^2x_3 - x_2x_3^2)$$

$$\mathrm{Betti}(A) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & 4 & 4 & 1 \\ 2 & . & 2 & 4 & 2 \end{array}.$$

$A = R/(\mathbf{I}(X), L)$ has the WLP.

Maximal is key

Example (Abdullah-Schenck, 2024)

$$I = (x_4^2, x_3x_4, x_3^3, x_2x_3^2 - x_2^2x_4, \\ x_1x_3^2 - x_1x_2x_4 + x_2^2x_4, x_2^2x_3, x_2^3, \\ x_1^3x_4 - x_1^2x_2x_4 + x_1x_2^2x_4, x_1^3x_3, \\ x_1^3x_2 - x_1^2x_2^2, x_1^4)$$

$$J = (x_1x_4, x_1^2, x_3x_4^2, x_2x_4^2, x_2^2x_4, x_1x_3^2, \\ x_1x_2^2 - x_3^2x_4, x_3^4, x_2x_3^3 - x_4^4, \\ x_2^2x_3^2, x_2^4)$$

	0	1	2	3	4
0	1	.	.	.	.
1	.	2	1	.	.
2	.	5	9	4	. .
3	.	4	9	5	.
4	.	.	2	1	.
5	.	.	.	.	1

Has the WLP

	0	1	2	3	4
0	1	.	.	.	.
1	.	2	1	.	.
2	.	5	9	4	. .
3	.	4	9	5	.
4	.	.	2	1	.
5	.	.	.	.	1

Does not have the WLP

Future work

(1) Can we do better than a *Koszul* tail?

(2) Is there a nice "Boij-Söderberg"-theoretic framework for this?

(3) Characterize Stanley-Reisner rings with $A = I_\Delta + \dots$ having (maximal) Koszul tails.

Future work

(1) Can we do better than a *Koszul* tail?

(2) Is there a "nice" underlying Boij-Söderberg framework for this?

(3) Characterize Stanley-Reisner rings with $A = I_\Delta + \ldots$ having (maximal) Koszul tails.

(4) Visit Sweden!