The Graded Möbius Algebra of a Matroid

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Organizers: John Cobb, Sean Grate, Michael Brown

https://johndcobb.github.io/organizing/M2atAuburn/

Objectives

- Develop new software for Macaulay2 and its packages
- Form new collaborations between different mathematical areas
- Train students through hands-on learning and coding

Group Leader	**Focus**
Keller VandeBogert	*Varieties* package
Juliette Bruce	toric geometry
Jay Yang	TBD
Thomas Brazelton	homotopy theory
Jordy Garcia-Lopez	Grothendieck-Witt computations
Sean Grate	matroids...?

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The Crew

Sean Grate (Me)

Jason McCullough

Free Resolutions

Definition

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous ideal, and consider the graded minimal free resolution of $R = S/I$:

$$0 \leftarrow R \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n+1} \leftarrow 0,$$

where $F_i \cong \bigoplus_j S(-i-j)^{\beta_{i,i+j}}$.

- $\beta_{i,i+j}$ are the graded Betti numbers
- $\beta_i = \sum_j \beta_{i,i+j}$ are the total Betti numbers

Example

$S = \mathbb{K}[x, y, z]$ and $I =< x^2, y^2, z^3 >$

$$0 \leftarrow S/I \leftarrow S \xleftarrow{\begin{pmatrix} x^2 & y^2 & z^3 \end{pmatrix}} \begin{matrix} S(-2)^2 \\ \oplus \\ S(-3) \end{matrix} \xleftarrow{\begin{pmatrix} -y^2 & -z^3 & 0 \\ x^2 & 0 & -z^3 \\ 0 & x^2 & y^2 \end{pmatrix}} \begin{matrix} S(-4) \\ \oplus \\ S(-5)^2 \end{matrix} \xleftarrow{\begin{pmatrix} z^3 \\ -y^2 \\ x^2 \end{pmatrix}} S(-7) \leftarrow 0$$

	0	1	2	3
0	1	.	.	.
1	.	2	.	.
2	.	1	1	.
3	.	.	2	.
4	.	.	.	1

Example

$S = \mathbb{K}[x, y, z]$ and $I =< x^2, y^2, z^3 >$

$$0 \leftarrow S/I \leftarrow S \leftarrow \begin{matrix} S(-(1+1))^2 \\ \oplus \\ S(-(1+2)) \end{matrix} \leftarrow \begin{matrix} S(-(2+2)) \\ \oplus \\ S(-(2+3))^2 \end{matrix} \leftarrow S(-(3+4)) \leftarrow 0$$

	0	1	2	3
0	1	.	.	.
1	.	2	.	.
2	.	1	1	.
3	.	.	2	.
4	.	.	.	1

Invariants from the Betti Table

	0	1	2	3
0	1	.	.	.
1	.	2	.	.
2	.	1	1	.
3	.	.	2	.
4	.	.	.	1

- Castelnuovo-Mumford regularity = height of the Betti table

$$\mathrm{reg}(S/I) = 4$$

- Projective dimension = length of the Betti table

$$\mathrm{pdim}(S/I) = 3$$

- Hilbert series

$$\mathrm{H}_{S/I}(t) = \frac{1}{(1-t)^3} - \left(\frac{2t^2}{(1-t)^3} + \frac{t^3}{(1-t)^3}\right) + \cdots = 1 + 3t + 4t^2 + \ldots$$

Koszul Algebras

R a (standard-graded) $\mathbb{K}$-algebra
$\mathfrak{m} = \bigoplus_{i \geq 1} R_i$

Definition

R is Koszul if $R/\mathfrak{m} \cong \mathbb{K}$ has a linear free resolution over R.

Example

$R = \mathbb{K}[x, y, z]$

$$0 \leftarrow R/\mathfrak{m} \leftarrow R \xleftarrow{(x\ y\ z)} R(-1)^3 \xleftarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} R(-2)^3 \xleftarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R(-3) \leftarrow 0$$

	0	1	2	3
0	1	3	3	1

More Examples

- Quadratic complete intersections (Tate, 1957)
- Universal enveloping algebras (Priddy, 1970)
- Quotients by quadratic monomial ideals (Fröberg, 1975)
- Sufficiently high Veronese subalgebras of a standard-graded $\mathbb{K}$-algebra (Backelin, 1986)
- Coordinate rings of Grassmannians (Kempf, '90) and general canonical curves (Vishik-Finkelberg, 1993)

When is a $\mathbb{K}$-algebra Koszul?

Theorem (Conca-Rossi-Valla, 2001)

Suppose R is a graded Gorenstein $\mathbb{K}$-algebra presented by quadrics. If $\operatorname{reg}(R) \leq 2$ *or if* $\operatorname{reg}(R) = 3$ *and* $\operatorname{codim}(R) \leq 4$*, then R is Koszul.*

Theorem (Mastroeni-Schenck-Stillman, 2021/2023)

For all integers $c \geq r + 2 \geq 6$, there exists a graded, quadratic, Gorenstein $\mathbb{K}$-algebra R with $\operatorname{reg}(R) = r$ *and* $\operatorname{codim}(R) = c$ *that is not Koszul.*

Theorem (Roos, 1993)

Koszulness of a $\mathbb{K}$-algebra cannot definitively be checked by computing finitely many steps of the minimal free resolution.

Properties of Koszul Algebras

Definition

The Poincaré series $P_R(t)$ of R is the generating function for the total Betti numbers of $R/\mathfrak{m} \cong \mathbb{K}$.

Theorem

If $R = S/I$ is Koszul, then

- *I is generated by quadrics;*
- *$reg(R) \leq pdim(R)$;*
- *$\beta_{i,i+j} = 0$ for $j > i$;*
- *$\beta_{i,2i} \leq \binom{c}{i}$, where c is the number of quadrics;*
- *$P_R(t)H_R(-t) = 1$.*

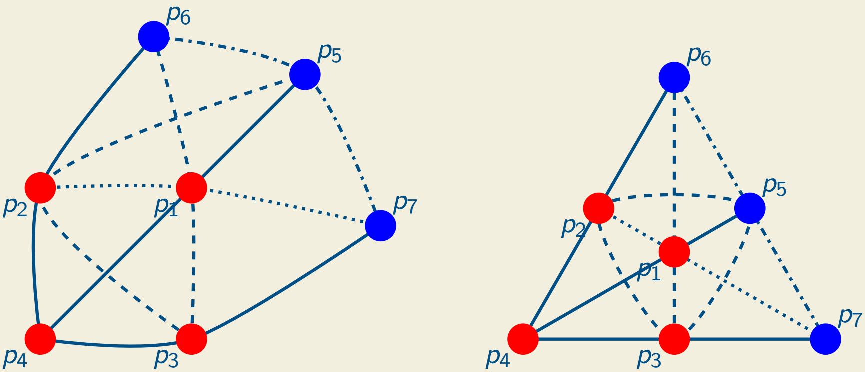
Proving Koszulness

- Construct a Kosul filtration.
  - A family of ideals closed under a certain colon operation (roughly).
  - Typically, the filtration is very long.
- Show R is G-quadratic.

$$\begin{cases} \text{quadaratic Gröbner basis} \\ \\ \text{graded Betti numbers} \\ \text{upper-semicontinuous} \end{cases} \implies R \text{ Koszul}$$

Projective Geometries

$$\begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$



Matroids Crash Course

Definition

A matroid $M = (E, \mathcal{I}$ consists of a ground set E and independent sets $\mathcal{I} \subseteq 2^E$ such that

(1) $\emptyset \in \mathcal{I}$;

(2) If $I' \subseteq I$ and $I \in \mathcal{I}$, then $I' \in \mathcal{I}$;

(3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

Example

- Vector spaces
- Cycles of a graph
- Hyperplane arrangements
- Boolean lattices

Matroids Crash Course

Let $M = (E, \mathcal{I})$ be a matroid.

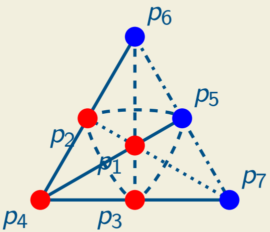
- Elements of $\mathcal{I}$ are called independent sets; maximal ones are called bases.
- Sets not in $\mathcal{I}$ are called dependent sets; minimal ones are called circuits.
- The rank of a set $F \subseteq E$ is the cardinality of a maximal independent set in F. The closure of a set $F \subseteq E$ is

$$\overline{F} := \{e \in E \ : \ \text{rank}(F \cup \{e\}) = \text{rank}(F)\}.$$

A set F such that $\overline{F} = F$ is called a flat.

$\mathbf{PG}(2,2) = F_7$

$$\begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$



Set	Type	Rank	Closure
$\{p_2, p_3, p_4\}$	basis	3	$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$
$\{p_2, p_3\}$	independent	2	$\{p_2, p_3, p_5\}$
$\{p_1, p_2, p_3, p_4\}$	circuit	3	$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$
$\{p_3, p_4, p_7\}$	dependent, flat	2	$\{p_3, p_4, p_7\}$

Geometric Lattices

Definition

A *lattice* *is a poset with well-defined join* $\vee$ *and meet* $\wedge$ *operations.*

Definition

A lattice is called *geometric* *if it is*

(1) atomic

(2) (upper) semimodular

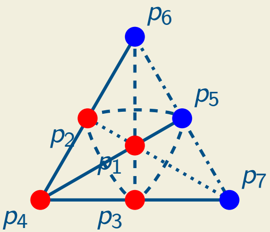
$$\mathrm{rank}(F \vee G) + \mathrm{rank}(F \wedge G) \leq \mathrm{rank}(F) + \mathrm{rank}(G)$$
$$\dim(U + V) + \dim(U \cap V) \leq \dim(U) + \dim(V)$$

Theorem

A lattice is geometric if and only if it is the lattice of flats of a matroid.

The Lattice of Flats

$$\begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\ \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$



$\#\mathcal{L}^i$		rank
1	1234567	3
7	127 136 145 235 246 347 567	2
	$\cdots$	
7	1 2 3 4 5 6 7	1
1	$\emptyset$	0

How do different ranks compare?

Theorem (Greene, 1970)

Let $\mathcal{L}$ be a geometric lattice with $\operatorname{rank}\mathcal{L} = n > 1$*, and with*

$$\mathcal{L}^k := \{F \in \mathcal{L} \ : \ \operatorname{rank}(F) = k\},$$

let $W_k = \left|\mathcal{L}^k\right|$. Then

(1) $W_1 \leq W_k$ *for all* $k = 2, \ldots, n-1$*; and*

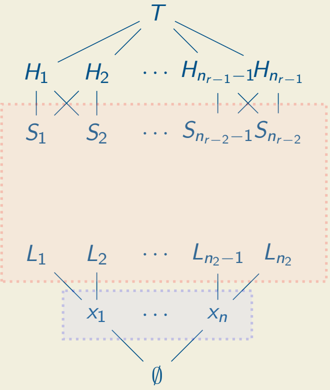
(2) $W_1 = W_k$ *if and only if* $k = n-1$ *and L is modular.*

$$\operatorname{rank}(F \vee G) + \operatorname{rank}(F \wedge G) = \operatorname{rank}(F) + \operatorname{rank}(G)$$

The Top-Heavy Conjecture

Theorem (Huh-Wang, 2017; Braden-Huh-Matherne-Proudfoot-Wang, 2020)

Let $\mathcal{L}$ be a geometric lattice of rank r. For any $k \leq j \leq r - k$, there is an injective map $\iota\colon \mathcal{L}^k \rightharpoonup \mathcal{L}^j$.



Graded Möbius Algebras

Definition

Let $\mathcal{M}$ be a simple matroid with lattice of flats $\mathcal{L}(M)$. The graded Möbius algebra of M is the standard graded $\mathbb{K}$-algebra

$$\mathrm{B}_M = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K} x_F,$$

with multiplication

$$x_F x_G = \begin{cases} x_{F \vee G} & \text{if } \operatorname{rank}(F \vee G) = \operatorname{rank}(F) + \operatorname{rank}(G), \\ 0 & \text{otherwise} \end{cases}.$$

Properties

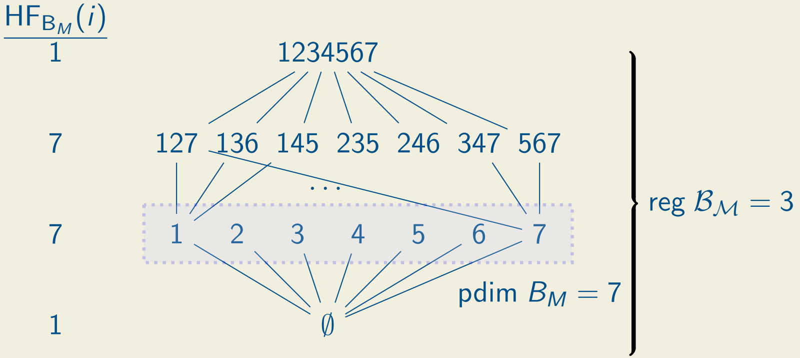
Fact

Let M be a simple matroid with lattice of flats $\mathcal{L}(M)$.

(1) The Hilbert function of B_M is the rank-generating function of $\mathcal{L}(M)$

(2) $\operatorname{reg} \mathrm{B}_M = \operatorname{rank} M = \operatorname{rank} \mathcal{L}(M)$

(3) $\operatorname{pdim} \mathrm{B}_M = |E(M)|$



When is B_M Koszul?

Theorem (Laclair-Mastroeni-McCullough-Peeva, 2025)

Let $\mathcal{M}$ be a simple matroid. There is a presentation of B_M:

$$\mathrm{B}_M \cong \frac{\mathbb{K}[y_i : \ i \in E(M)]}{(y_i^2 : \ i \in E) + (y_{C\setminus i} - y_{C\setminus j} : \ C \in \mathcal{C}(M))}$$

Theorem (Laclair-Mastroeni-McCullough-Peeva, 2025)

Let M be the cycle matroid of a graph G. Then B_M is Koszul if and only if G is strongly chordal.

Orlik-Solomon + Graded Möbius Algebra Parallels

Let $M = M(G)$ be the cycle matroid of a graph G, and let A_M be its Orlik-Solomon algebra and B_M its graded Möbius algebra.

$$\begin{tikzcd}
A_G \text{ G-quadratic} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & M \text{ supersolvable} \arrow[dd, Leftrightarrow] & \mathrm{B}_M \text{ G-quadratic} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & M \text{ strongly T-chordal} \arrow[d, Leftrightarrow] \\
A_G \text{ Koszul} \arrow[d, Leftrightarrow] & & \mathrm{B}_M \text{ Koszul} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & G \text{ strongly chordal} \\
A_G \text{ qaudratic} \arrow[r, Leftrightarrow] & G \text{ chordal} \arrow[r, Leftrightarrow] & \mathrm{B}_M \text{ quadratic} &
\end{tikzcd}$$

Projective Geometries

Theorem (Greene, 1970; Maeno-Numata, 2016)

Let M be a simple matroid and $\mathcal{L}(M)$ be its lattice of flats. The following are equivalent:

(1) $\mathcal{L}(M)$ is modular.

(2) The rank-generating function of $\mathcal{L}(M)$ is symmetric.

(3) The Hilbert function of B_M is symmetric.

(4) B_M is Gorenstein.

$$\mathrm{B}_M \cong \frac{\mathbb{K}[y_i : i \in E(M)]}{Ann\left(\sum_{B \in \mathcal{B}(M)} y_B\right)}$$

Projective Geometries

Theorem (Grate-McCullough, 2026+)

If M is a projective geometry, then B_M is Koszul.

$$
\begin{tikzcd}
M = \mathrm{PG}(d,q) \arrow[r, Rightarrow, "{[\mathrm{GM26+}]}"'] \arrow[d, Rightarrow, "{[\mathrm{GM26+}]}"] & M \text{ chordal} \arrow[r, Rightarrow, "\text{Proposition 3.5(a)}", "{[\mathrm{LMMP25}]}"'] & \mathrm{B}_M \text{ quadratic} \\
M \text{ strongly T-chordal} \arrow[r, Leftrightarrow, "\text{Theorem 3.9}", "{[\mathrm{LMMP25}]}"'] & \mathrm{B}_M \text{ G-quadratic} \arrow[r, Rightarrow] & \mathrm{B}_M \text{ Koszul} \arrow[u, Rightarrow]
\end{tikzcd}
$$

Affine Geometries

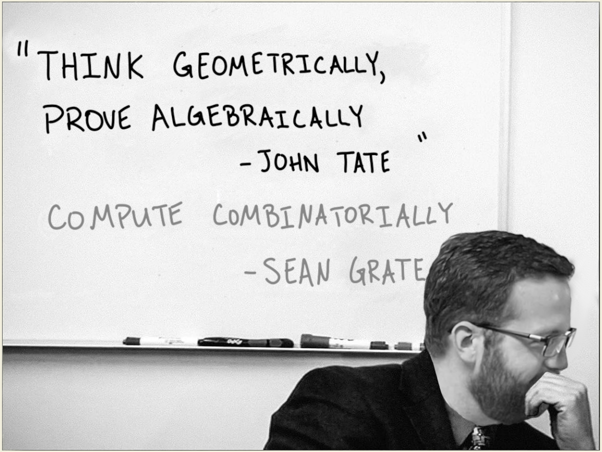
Fact

Affine geometries are obtained from projective geometries by deleting the points on the hyperplane at infinity.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
M = \mathrm{AG}(d,q) \arrow[r, Rightarrow, "\mathrm{char}(\mathbb{K})\neq 2", "{[\mathrm{GM26+}]}"'] \arrow[rr, Rightarrow, bend right, "q\geq 3", "{[\mathrm{GM26+}]}"'] \arrow[d, Rightarrow, "\text{Conjectured}"', "{[\mathrm{GM26+}]}"] \arrow[dr, Rightarrow, bend right, "M=\mathrm{AG}(2,3)", "{[\mathrm{LMMP25}]}"'] \arrow[drr, Rightarrow, bend right, "\text{Conjectured } (q\geq 3)", "{[\mathrm{GM26+}]}"'] & M \text{ chordal} \arrow[r, Rightarrow, "\text{Proposition 3.5(a)}", "{[\mathrm{LMMP25}]}"'] & \mathrm{B}_M \text{ quadratic} \\
M \text{ strongly T-chordal} \arrow[r, Leftrightarrow, "\text{Theorem 3.9}", "{[\mathrm{LMMP25}]}"'] & \mathrm{B}_M \text{ G-quadratic} \arrow[r, Rightarrow] & \mathrm{B}_M \text{ Koszul} \arrow[u, Rightarrow]
\end{tikzcd}
$$

Future Directions

(1) Can we (combinatorially) compute a graded minimal free resolution of B_M?



Theorem (FGK+, 2026)

We can compute the Betti numbers for uniform matroids $U_{r,n}$.

Example

The Betti table for $B_{U_{6,8}}$ is

	0	1	2	3	4	5	6	7	8
0	1	.	.	.	.	.	.	.	.
1	.	8	.	.	.	.	.	.	.
2	.	.	28	.	.	.	.	.	.
3	.	.	.	56	.	.	.	.	.
4	.	.	.	.	70	.	.	.	.
5	.	27	208	693	1296	1526	1008	350	48
6	.	.	.	.	.	.	.	.	1

Example

Let $M = \mathrm{PG}(2,2)$. It is a rank 3 matroid on 7 elements with rank generating function $1 + 7t + 7t^2 + t^3$. The Betti table for B_M is

	0	1	2	3	4	5	6	7
0	1	.	.	.	.	.	.	.
1	.	21	64	70	14	.	.	.
2	.	.	.	14	70	64	21	.
3	.	.	.	.	.	.	.	1

.

Future Directions

(2) What other matroids have Koszul graded Möbius algebras?

Matroid	B_M **Koszul?**
M(G)	G is strongly chordal
PG(n, q)	$n \geq 2$
AG(n, q)	$n \geq 2$, $q \geq 3$ (conjectured)
Sylvester	not in general
Lattice Path	???
Transversal	???
(Sparse) Paving	???
Dowling geometry	???

Thank you!